HISTORY-DEPENDENT QUASIVARIATIONAL INEQUALITIES WITH APPLICATIONS IN CONTACT MECHANICS

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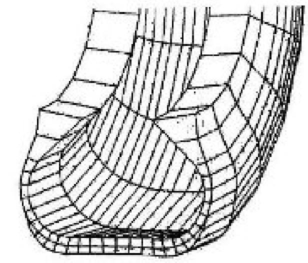


FIG. 5.20 – Écrasement statique - $F = 6$ tonnes

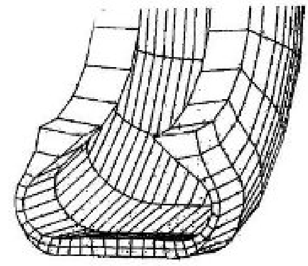


FIG. 5.21 – Écrasement statique - $F = 10$ tonnes

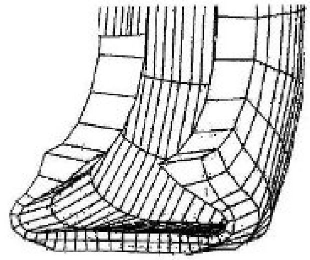
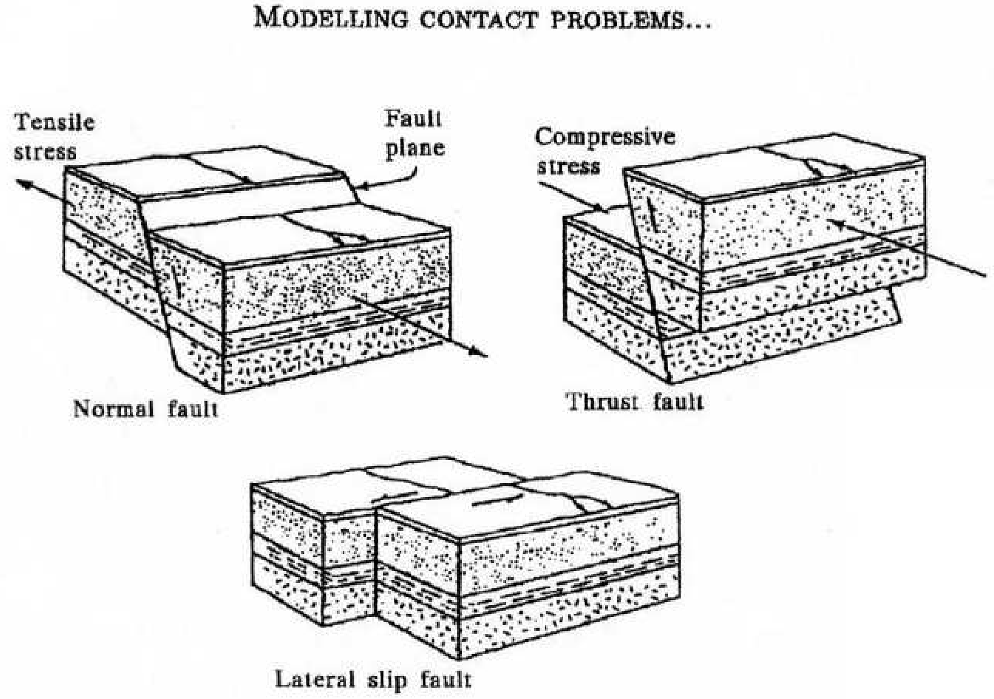
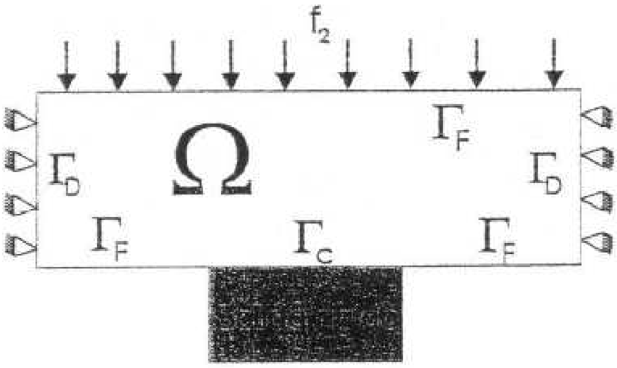
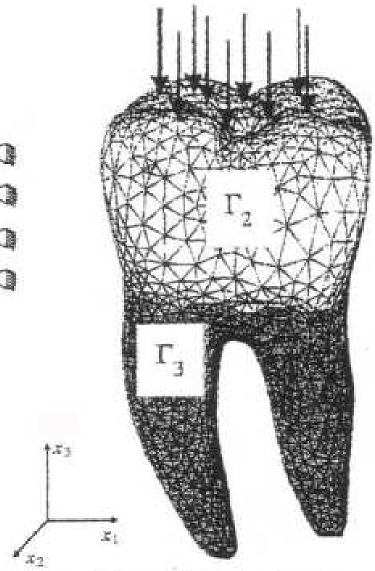


FIG. 5.23 – Écrasement statique - $F = 18$ tonnes

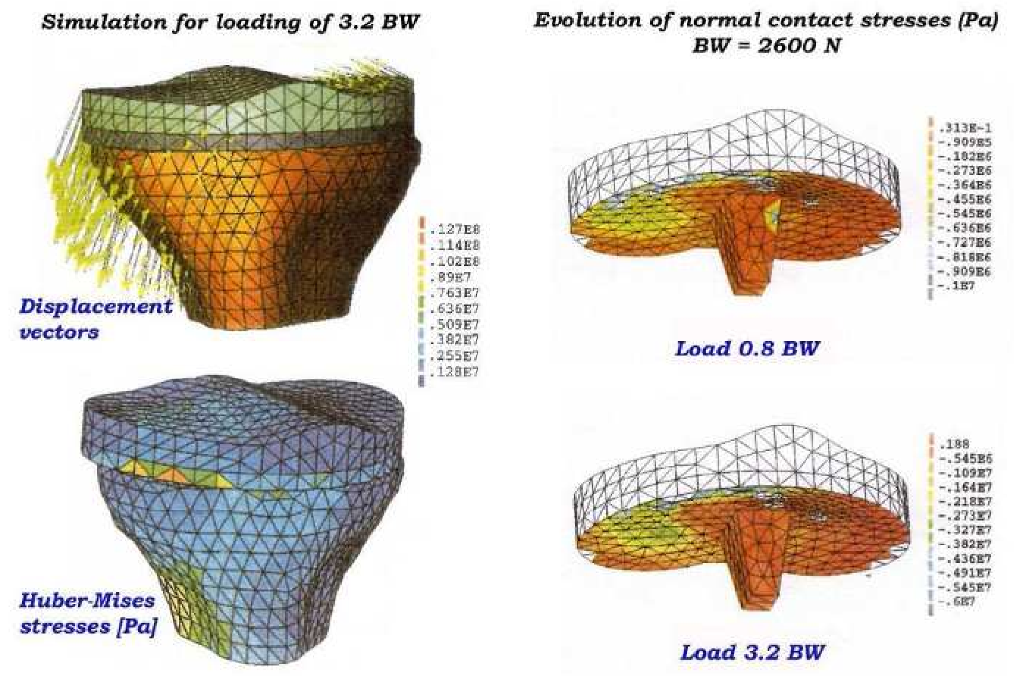

Tensile stress
Fault plane
Normal fault
Compressive stress
Thrust fault
Lateral slip fault



Three-dimensional contact problem with rigid body (lateral view)



Three-dimensional example : clenching task with the right lower first molar



Simulation for loading of 3.2 BW
Evolution of normal contact stresses (Pa)
BW = 2600 N
.313E-1
-.909E5
-.182E6
-.273E6
-.364E6
-.455E6
-.545E6
-.636E6
-.727E6
-.818E6
-.909E6
-.1E7
.127E8
.114E8
.102E8
.89E7
.763E7
.636E7
.509E7
.382E7
.255E7
.128E7
Displacement vectors
Load 0.8 BW
.188
-.545E6
-.109E7
-.164E7
-.218E7
-.273E7
-.327E7
-.382E7
-.436E7
-.491E7
-.545E7
-.6E7
Huber-Mises stresses [Pa]
Load 3.2 BW

I. INTRODUCTION

Main steps in the study of contact problems :

– **modelling** (constitutive law, contact boundary conditions, assumptions on external forces,...) ;

– **variational analysis** (variational formulation, existence and uniqueness results, properties of the solution,...) ;

– **numerical analysis** (analysis of semi-discrete and fully discrete schemes, error estimates,...) ;

– **numerical simulations**.

Constitutive law

- elastic (linear, nonlinear) ;
- viscoelastic (with short memory, with long memory) ;
- viscoplastic (with or without hardening, internal state variables).

Contact conditions

- unilateral (rigid foundation) ;
- with normal compliance (deformable foundation) ;
- with normal damped response (lubricated foundation).

Frictional conditions

- classical Coulomb's law of dry friction ;
- Tresca's law ;
- Stromberg's law (1995) ;
- slip, slip rate dependent friction laws ;
- total slip, total slip rate dependent friction laws.

Additional effects

- thermal effect ;
- piezoelectric effect ;
- damage ;
- adhesion ;
- wear.

Mathematical Theory of Contact Mechanics

It concerns the mathematical structures which underlie general contact problems with different constitutive laws, varied geometries, and different contact conditions.

Main feature

Cross fertilization between modelling and applications on the one-hand, and mathematical analysis on the other-hand.

Aim of this lecture

To study a contact problem within the MTCM and to provide an example of such cross fertilization.

Basic references

1. **A. Signorini**, *Sopra alcune questioni di elastostatica.* Atti della Società Italiana per il Progresso delle Scienze, 1933.

2. **G. Fichera**, *Problemi elastostatici con vincoli unilaterali. II. Problema di Signorini con ambique condizioni al contorno*, Mem. Accad. Naz. Lincei, S. VIII, Vol. VII, Sez. I, 5, pp. 91–140, 1964.

3. **G. Duvaut** and **J. L. Lions**, *Inequalities in Mechanics and Physics,* Springer-Verlag, Berlin, 1976.

4. **P. D. Panagiotopoulos**, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, 1985.

5. **N. Kikuchi** and **J. T. Oden**, *Contact Problems in Elasticity : A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia, 1988.

II. QUASIVARIATIONAL INEQUALITIES

Notation

$\mathbb{N}^*$ - set of positive integers ;

$\mathbb{R}_+$ - set of nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$;

$(X, (\cdot, \cdot)_X, \|\cdot\|_X)$ - real Hilbert space, $K \subset X$;

$(Y, \|\cdot\|_Y)$ - real normed space ;

$\mathcal{L}(X, Y)$, $C(\mathbb{R}_+; X)$, $C^1(\mathbb{R}_+; X)$ - standard notation ;

$C(\mathbb{R}_+; K)$, $C^1(\mathbb{R}_+; K)$ - set of continuous and continuously differentiable functions defined on $\mathbb{R}_+$ with values on K, respectively.

Problem 1. *Find* $u \in C(\mathbb{R}_+; K)$ *such that, for all* $t \in \mathbb{R}_+$,

$$
\begin{aligned}
(1) \quad & (Au(t), v - u(t))_X + \varphi(\mathcal{S}u(t), v) - \varphi(\mathcal{S}u(t), u(t)) \\
& + j(u(t), v) - j(u(t), u(t)) \geq (f(t), v - u(t))_X \quad \forall\, v \in K.
\end{aligned}
$$

Here :

$A : K \to X$,
$\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$, $\mathcal{S}u(t) = (\mathcal{S}u)(t)$ for all $t \in \mathbb{R}_+$,
$\varphi : Y \times K \to \mathbb{R}$,
$j : X \times K \to \mathbb{R}$,
$f : \mathbb{R}_+ \to X$.

Assumptions

(2) K is a closed, convex, nonempty subset of X.

$$(3)\quad \begin{cases} \text{(a) There existes } m > 0 \text{ such that} \\ \quad (Au_1 - Au_2, u_1 - u_2)_X \geq m\, \|u_1 - u_2\|_X^2 \\ \quad \forall\, u_1, u_2 \in K. \\ \text{(b) There exists } L > 0 \text{ such that} \\ \quad \|Au_1 - Au_2\|_X \leq L\, \|u_1 - u_2\|_X \\ \quad \forall\, u_1, u_2 \in K. \end{cases}$$

$$
(4) \quad \begin{cases} \text{(a) For all } y \in Y,\ \varphi(y,\cdot) \text{ is convex and l.s.c. on } K. \\ \text{(b) There exists } \alpha > 0 \text{ such that} \\ \qquad \varphi(y_1,u_2) - \varphi(y_1,u_1) + \varphi(y_2,u_1) - \varphi(y_2,u_2) \\ \qquad \le \alpha \, \|y_1 - y_2\|_Y \, \|u_1 - u_2\|_X \\ \qquad \forall\, y_1,\, y_2 \in Y,\ \forall\, u_1,\, u_2 \in K. \end{cases}
$$

$$
(5) \quad \begin{cases} \text{(a) For all } u \in X,\ j(u,\cdot) \text{ is convex and l.s.c. on } K. \\ \text{(b) There exists } \beta > 0 \text{ such that} \\ \qquad j(u_1,v_2) - j(u_1,v_1) + j(u_2,v_1) - j(u_2,v_2) \\ \qquad \le \beta \, \|u_1 - u_2\|_X \, \|v_1 - v_2\|_X \\ \qquad \forall\, u_1,\, u_2 \in X,\ \forall\, v_1,\, v_2 \in K. \end{cases}
$$

(6) $\beta < m$.

$$(7) \quad \begin{cases} \text{For all } n \in \mathbb{N}^* \text{ there exists } r_n > 0 \text{ such that} \\ \quad \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_Y \leq r_n \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X \, ds \\ \quad \forall\, u_1,\, u_2 \in C(\mathbb{R}_+; X),\ \forall\, t \in [0, n]. \end{cases}$$

(8) $f \in C(\mathbb{R}_+; X)$.

Remark 1. Condition (7) is satisfied for the integral operator and for the Volterra-type operators.

Theorem 1. *Assume that* (2)–(8) *hold. Then, the quasivariational inequality* (1) *has a unique solution* $u \in C(\mathbb{R}_+; K)$.

Proof. The proof is carried out in several steps. It is based on arguments of elliptic variational inequalities, monotonicity and fixed point. The crucial ingredient is the use of a new fixed point result obtained in

M. Sofonea, C. Avramescu and A. Matei, A Fixed point result with applications in the study of viscoplastic frictionless contact problems, *Communications on Pure and Applied Analysis*, **7** (2008), 645–658. □

Notation

$\mathcal{C}_+$ - set of des compact intervals included in $\mathbb{R}_+$. Moreover, for $1 \leq p \leq \infty$ and $k = 1, 2, \ldots$, denote

$$W^{k,p}_{loc}(\mathbb{R}_+, X) = \{ \ u : \mathbb{R}_+ \to X \ : \ u \in W^{k,p}(I, X) \ \ \forall \, I \in \mathcal{C}_+ \ \};$$

$$W^{k,p}(I, K) = \{ \ u : \mathbb{R}_+ \to K \ : \ u \in W^{k,p}(I, X) \ \};$$

$$W^{k,p}_{loc}(\mathbb{R}_+, K) = \{ \ u : \mathbb{R}_+ \to K \ : \ u \in W^{k,p}(I, K) \ \ \forall \, I \in \mathcal{C}_+ \ \}.$$

Theorem 2. *Assume that* (2)–(8) *hold and, moreover, assume that* Y *is a reflexive Banach space. Assume in addition that there exists* $p \in [1, \infty]$ *such that*

$$f \in W^{1,p}_{loc}(\mathbb{R}_+, X) \quad \text{and} \quad \mathcal{S}v \in W^{1,p}_{loc}(\mathbb{R}_+, Y) \quad \forall\, v \in C(\mathbb{R}_+; X).$$

Then, the solution of the quasivariational inequality (1) *has the regularity* $u \in W^{1,p}_{loc}(\mathbb{R}_+, K)$.

Proof. Let $I \in \mathcal{C}_+$. We prove that $u : I \to K$ is absolutely continuous and, moreover,

$$\|\dot{u}(t)\|_X \leq c\left(\left\|\frac{d}{dt}\big(\mathcal{S}u(t)\big)\right\|_Y + \|\dot{f}(t)\|_X\right) \quad \forall\, t \in I.$$

We conclude that $\dot{u} \in L^p(I; X)$ and, therefore, $u \in W^{1,p}_{loc}(\mathbb{R}_+, K)$.□

Numerical analysis

In the study of Problem 1 we obtained results concerning :

- existence and uniqueness of the solution for semi-discrete and fully-discrete approximation scheme ;
- error estimates for semi-discrete and fully-discrete approximation scheme.

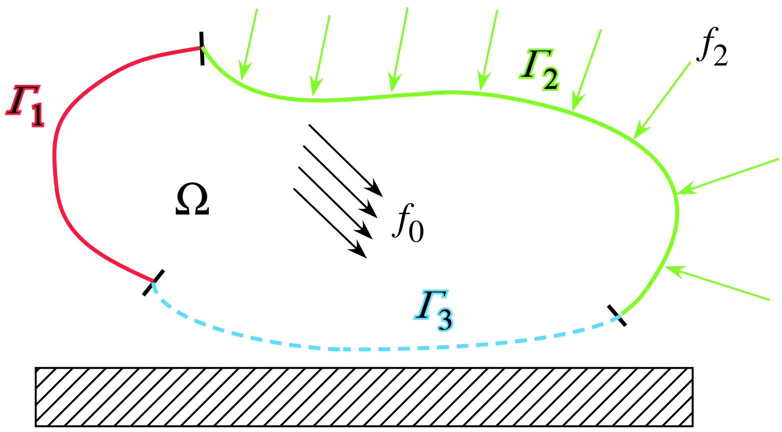


FIG. 1 – Physical setting

III. A FRICTIONAL CONTACT PROBLEM

Notation

Ω - bounded domain of $\mathbb{R}^d$ $(d = 2, 3)$;

Γ - boundary of Ω;

Γ_1, Γ_2, Γ_3 - partition of Γ such that $meas\ \Gamma_1 > 0$;

$\boldsymbol{\nu}$ - unit outward normal on Γ;

$\mathbb{S}^d$ - space of second order symmetric tensors on $\mathbb{R}^d$;

"$\cdot$", $\|\cdot\|$ - inner product and Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$.

$\boldsymbol{\sigma}$ - stress tensor ;

$\boldsymbol{u}$ - displacement field ;

$\boldsymbol{\varepsilon}$ - the deformation operator :

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i});$$

Div - the divergence operator : $\mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j})$;

v_ν, $\boldsymbol{v}_\tau$ - *normal* and *tangential* components of $\boldsymbol{v}$ on Γ :

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu};$$

σ_ν, $\boldsymbol{\sigma}_\tau$ - *normal* and *tangential* components of $\boldsymbol{\sigma}$ on Γ :

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu};$$

Problem $\mathcal{P}$. *Find the displacement $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and the stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ such that, for all $t > 0$,*

$$
\begin{aligned}
\boldsymbol{\sigma}(t) &= \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) && \text{in } \Omega,\\
\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) &= \boldsymbol{0} && \text{in } \Omega,\\
\boldsymbol{u}(t) &= \boldsymbol{0} && \text{on } \Gamma_1,\\
\boldsymbol{\sigma}(t)\boldsymbol{\nu} &= \boldsymbol{f}_2(t) && \text{on } \Gamma_2,\\
u_\nu(t) &= 0 && \text{on } \Gamma_3,\\
&\left.\begin{aligned}
&\|\boldsymbol{\sigma}_\tau(t)\| \le g(\|\dot{\boldsymbol{u}}_\tau(t)\|),\\
&-\boldsymbol{\sigma}_\tau(t) = g(\|\dot{\boldsymbol{u}}_\tau(t)\|)\,\frac{\dot{\boldsymbol{u}}_\tau(t)}{\|\dot{\boldsymbol{u}}_\tau(t)\|} \quad \text{if} \quad \dot{\boldsymbol{u}}_\tau(t) \neq \boldsymbol{0}
\end{aligned}\right\} && \text{on } \Gamma_3,\\
\boldsymbol{u}(0) &= \boldsymbol{u}_0 && \text{in } \Omega.
\end{aligned}
$$

Notation

$V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d : \boldsymbol{v} = \mathbf{0} \ \text{ a.e. on } \Gamma_1,\ v_\nu = 0 \ \text{ a.e. on } \Gamma_3 \,\}\,;$

$Q = \{\ \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d\times d} : \tau_{ij} = \tau_{ji},\ 1 \le i, j \le d\ \}\,;$

Inner products :

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx;$$

Associated norms : $\|\cdot\|_V,\ \|\cdot\|_Q.$

Assumptions

$$
(11)\quad \left\{
\begin{array}{l}
\text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\
\qquad \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\qquad (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}}\, \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } Q.
\end{array}
\right.
$$

$$
(12)\quad \begin{cases}
\text{(a) } \mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\
\qquad \|\mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{B}}\, \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } Q.
\end{cases}
$$

$$
(13)\quad \boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \qquad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d).
$$

$$
(14)\quad \begin{cases}
\text{(a) } g : \Gamma_3 \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+. \\
\text{(b) There exists } L_g > 0 \text{ such that} \\
\qquad |g(\boldsymbol{x}, r_1) - g(\boldsymbol{x}, r_2)| \le L_g \, |r_1 - r_2| \\
\qquad\quad \forall\, r_1,\ r_2 \in \mathbb{R},\ \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto g(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \\
\qquad \text{for any } r \in \mathbb{R}. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto g(\boldsymbol{x}, 0) \text{ belongs to } L^2(\Gamma_3).
\end{cases}
$$

$$
(15)\quad \boldsymbol{u}_0 \in V.
$$

Let $A : V \to V$, $\varphi : V \times V \to \mathbb{R}$, $j : V \times V \to \mathbb{R}$, $\boldsymbol{f} : \mathbb{R}_+ \to V$ and $\mathcal{S} : C(\mathbb{R}_+, V) \to C(\mathbb{R}_+, V)$ be defined by

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \forall\, \boldsymbol{u},\ \boldsymbol{v} \in V,$$

$$\varphi(\boldsymbol{u}, \boldsymbol{v}) = (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V,$$

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} g(\|\boldsymbol{u}_\tau(t)\|)\, \|\boldsymbol{v}_\tau\|\, da \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V,$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+,$$

$$\mathcal{S}\boldsymbol{v}(t) = \int_0^t \boldsymbol{v}(s)\, ds + \boldsymbol{u}_0 \qquad \forall\, \boldsymbol{v} \in C(\mathbb{R}_+, V),\ t \in \mathbb{R}_+.$$

Problem $\mathcal{P}_V$. *Find a velocity field $\boldsymbol{w} : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$,*

$$(A\boldsymbol{w}(t), \boldsymbol{v} - \boldsymbol{w}(t))_V + \varphi(\mathcal{S}\boldsymbol{w}(t), \boldsymbol{v}) - \varphi(\mathcal{S}\boldsymbol{w}(t), \boldsymbol{w}(t))$$
$$+j(\boldsymbol{w}(t), \boldsymbol{v}) - j(\boldsymbol{w}(t), \boldsymbol{w}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{w}(t))_V \;\forall \boldsymbol{v} \in V.$$

Theorem 3. *Assume that* (11)–(15) *hold. Then, there exists $L_0 > 0$ which depends only on Ω, Γ_1, Γ_3 and $\mathcal{A}$ such that Problem $\mathcal{P}_V$ has a unique solution $\boldsymbol{w} \in C(\mathbb{R}_+, V)$, if $L_g < L_0$. Moreover, if there exists $p \in [1, \infty]$ such that*

$$\text{(16)} \quad \boldsymbol{f}_0 \in W^{1,p}_{loc}(\mathbb{R}_+, L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in W^{1,p}_{loc}(\mathbb{R}_+, L^2(\Gamma_2)^d),$$

then the solution satisfies $\boldsymbol{w} \in W^{1,p}_{loc}(\mathbb{R}_+; V)$.

Proof. Problem $\mathcal{P}_V$ represents a quasivariational inequality of the form (1) in which $X = Y = K = V$. We prove that

$$j(\boldsymbol{u}_1, \boldsymbol{v}_2) - j(\boldsymbol{u}_1, \boldsymbol{v}_1) + j(\boldsymbol{u}_2, \boldsymbol{v}_1) - j(\boldsymbol{u}_2, \boldsymbol{v}_2) \\ \leq c_0^2 L_g \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \, \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V \quad \forall \, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V.$$

where c_0 depends only on Ω, Γ_1 and Γ_3. On the other hand, A satisfies condition (3) with $m = m_{\mathcal{A}}$. We use Theorem 1 to see that Problem $\mathcal{P}_V$ has a unique solution $\boldsymbol{w} \in C(\mathbb{R}_+, W)$, if $c_0^2 L_p < m_{\mathcal{A}}$. Therefore, we may take $L_0 = m_{\mathcal{A}}/c_0^2$.

Finally, we note that assumption (16) implies that $\boldsymbol{f} \in W^{1,p}_{loc}(\mathbb{R}_+; V)$. Therefore, by Theorem 2 we deduce that if (16) holds then $\boldsymbol{w} \in W^{1,p}_{loc}(\mathbb{R}_+; V)$, which completes the proof. $\square$

Remark 2. Let $\boldsymbol{w}$ denote a solution of Problem $\mathcal{P}_V$ and denote by $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ the functions defined by

$$\boldsymbol{u} = \mathcal{S}\boldsymbol{w}, \qquad \boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}).$$

Then, the couple $(\boldsymbol{u}, \boldsymbol{\sigma})$ is called a *weak solution* of the frictional contact problem $\mathcal{P}$. It follows that, under the assumptions of Theorem 3, the contact problem $\mathcal{P}$ has a unique weak solution, which satisfies

$$\boldsymbol{u} \in C^1(\mathbb{R}_+, V), \qquad \boldsymbol{\sigma} \in C(\mathbb{R}_+, Q).$$

In addition, if (16) holds then

$$\boldsymbol{u} \in W^{2,p}_{loc}(\mathbb{R}_+; V), \qquad \boldsymbol{\sigma} \in W^{1,p}_{loc}(\mathbb{R}_+; Q).$$

Numerical approximation

Notation :

1) Ω - polygon or a polyhedron ;

2) V^h - a the finite elements space of piecewise linear functions corresponding to a regular family of triangulation of Ω, compatible with the boundary decomposition ;

3) $k > 0$ - time step ;

4) $N \in \mathbb{N}^*, \quad t_n = nk, \quad \boldsymbol{f}_n = \boldsymbol{f}(t_n) \quad \text{for all} \quad 0 \leq n \leq N$;

3) $\mathcal{S}_n^{kh}\boldsymbol{w}^{kh} = k\sum_{j=0}^{n}{}' \boldsymbol{w}_j^{kh} + \boldsymbol{u}_0^h$, where a prime indicates the first and last terms in the summation are to be halved;

4) $\boldsymbol{u}_0^h \in V^h$ - a finite element approximation of $\boldsymbol{u}_0$.

Problem $\mathcal{P}_V^{kh}$. *Find the discrete velocity field* $\boldsymbol{w}^{kh} = \{\boldsymbol{w}_n^{kh}\}_{n\geq 0} \subset V^h$ *such that*

$$
\begin{aligned}
&(A\boldsymbol{w}_n^{kh}, \boldsymbol{v}^h - \boldsymbol{w}_n^{kh})_V + \varphi(\mathcal{S}_n^{kh}\boldsymbol{w}^{kh}, \boldsymbol{v}^h) - \varphi(\mathcal{S}_n^{kh}\boldsymbol{w}^{kh}, \boldsymbol{w}_n^{kh}) \\
&\quad + j(\boldsymbol{w}_n^{kh}, \boldsymbol{v}^h) - j(\boldsymbol{w}_n^{kh}, \boldsymbol{w}_n^{kh}) \geq (\boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{w}_n^{kh})_V \quad \forall\, \boldsymbol{v}^h \in V^h.
\end{aligned}
$$

Main results

1) existence and uniqueness of the discrete solution under assumption of Theorem 3;

2) error estimate of the form

$$\max_{0 \le n \le N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_V \le c\left(h + k^2\right),$$

provided that k is sufficiently small, under additional regularity of the solution.

Remark 3. The fully discrete approximation of the displacement field $\boldsymbol{u}$ of the frictional contact problem $\mathcal{P}$, denoted $\{\boldsymbol{u}_n^{kh}\}_{n\geq 0}$, is given by

$$\boldsymbol{u}_n^{kh} = k \sum_{j=0}^{n}{}' \boldsymbol{w}_j^{kh} + \boldsymbol{u}_0^h.$$

Then, an estimate of the form

$$\max_{0\leq n\leq N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{kh}\|_V \leq c\left(h + k^2\right)$$

was obtained.

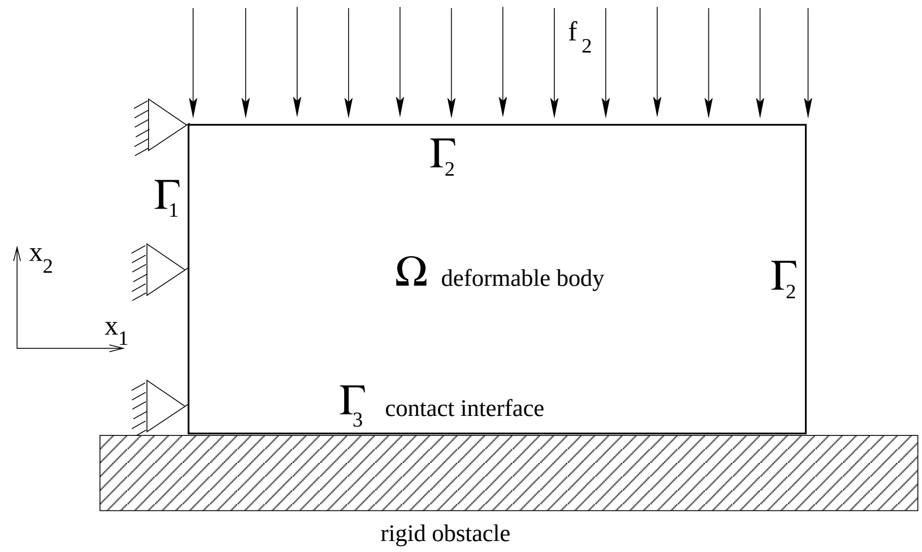


Fig. 2 – Initial configuration of the two-dimensional example.

Numerical example

$\Omega = (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$ with $L_1,\ L_2 > 0$,

$(\mathcal{A}\boldsymbol{\tau})_{\alpha\beta} = \mu_1(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \mu_2\tau_{\alpha\beta}$,

$(\mathcal{B}\boldsymbol{\tau})_{\alpha\beta} = \frac{E\kappa}{1-\kappa^2}(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \frac{E}{1+\kappa}\tau_{\alpha\beta}$,

$1 \leq \alpha, \beta \leq 2$, $\forall\, \boldsymbol{\tau} \in \mathbb{S}^2$, where μ_1 and μ_2 are viscosity constants, E and κ are Young's modulus and Poisson's ratio of the material, and $\delta_{\alpha\beta}$ denotes the Kronecker symbol.

$g(\|\dot{\boldsymbol{u}}_\tau\|) = [(a - b) \times e^{-\alpha\|\dot{\boldsymbol{u}}_\tau\|} + b]$ with $a, b, \alpha > 0$, $a \geq b$.

For computation we have used the following data :

$$L_1 = 1\,m, \quad L_2 = 0.5\,m,$$

$$\mu_1 = 0.05 N/m, \quad \mu_2 = 0.1 N/m, \quad E = 1 N/m, \quad \kappa = 0.3,$$

$$\boldsymbol{f}_0 = (0,0) N/m^2, \quad \boldsymbol{f}_2 = \begin{cases} (0,0)\, N/m & \text{on } \{1\} \times [0,0.5], \\ (0,-0.3)\, N/m & \text{on } [0,1] \times \{0.5\}, \end{cases}$$

$$a = 0.003, \quad b = 0.001, \quad \alpha = 100, \quad \boldsymbol{u}_0 = \mathbf{0}\,m.$$

We use :

- uniform meshes, obtained by dividing the interval [0,1] into $1/h$ equal parts in both x_1 and x_2 directions ;

- uniform partitions of the time interval $[0, 1]$;

- the numerical solution corresponding to $h = 1/256$ and $k = 1/256$ as the “exact” solution in computing the errors of the numerical solutions. This discretization corresponds to a problem with 132354 degres of freedom and the simulation runs in around 109 hours of CPU time.

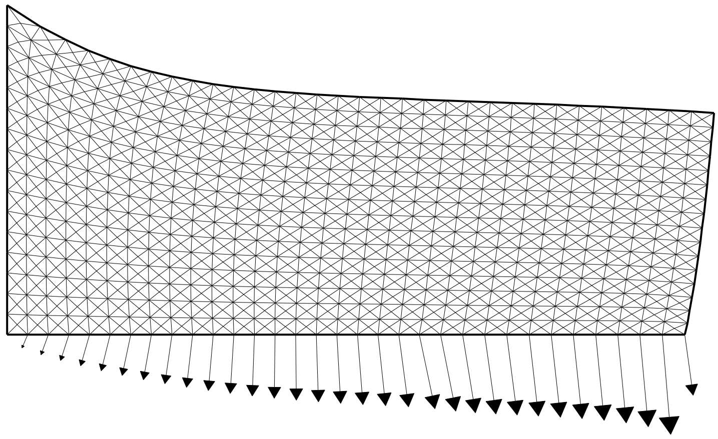


FIG. 3 – Deformed mesh and contact interface forces on Γ_3.

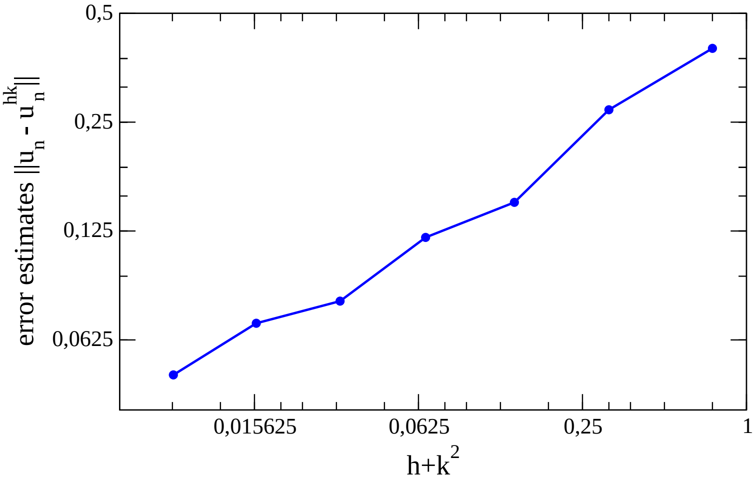


FIG. 4 – Estimated numerical errors.

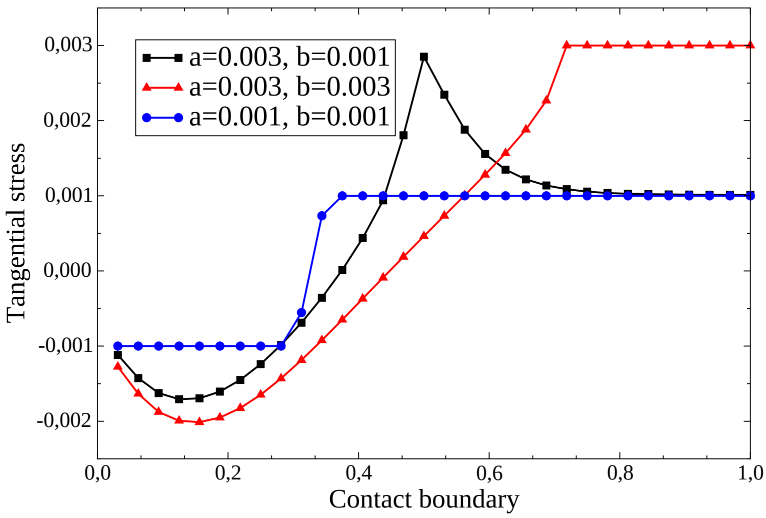


FIG. 5 – Tangential stresses on Γ_3.

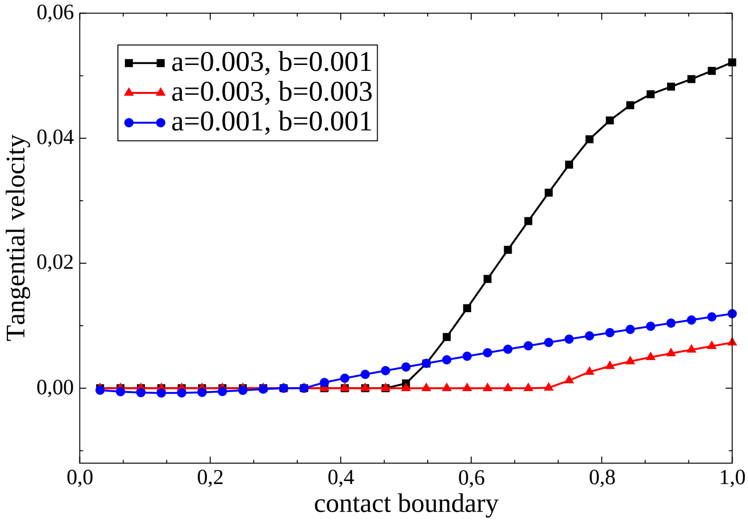


FIG. 6 – Tangential velocities on Γ_3.

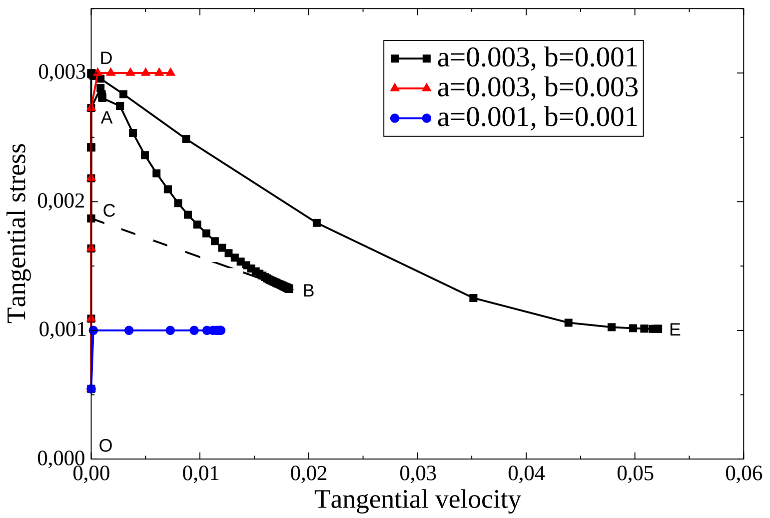


FIG. 7 – Tangential stresses versus tangential displacements at the node of coordinates (1.,0.).

Open questions

a) Is the smallness assumption $L_g < L_0$ an intrinsic feature of the model $\mathcal{P}_V$ or it represents only a limitation of our mathematical approach?

b) Find a reliable estimate of the critical value L_0.

c) Regularity results for the solution of Problems $\mathcal{P}_V$.

d) Extension of the results in the case of unilateral contact conditions.

e) Extension of the results in the dynamic case.

IV. REFERENCES

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