Some classes of stochastic processes in finance

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1 Financial markets

Stochastic processes (in particular *martingales, stochastic calculus* and *stochastic differential equations*): *fundamental tool in problems from financial mathematics to model the trading of risky assets (securities) in discrete and continuous time*

M. Black and F. Scholes (1973): the most famous financial model in continuous time (They were awarded with the Nobel price in economy).

They derived an explicit formula (*option pricing*) for the price of an European call on a stock paying no dividends and constructed a perfect *hedgeable replicating strategy.*

A brief description of the Black-Scholes model (*European call option*): Assume that at time $t = 0$ we (*the buyer, holder*) sign a contract with the *seller* (*writer*) which gives us the right to by (but not the obligation), at a specified time T (*maturity, expiration time*) one share of stock at a specified price K (*exercise, strike price*).

At maturity , if the price S_T of the stock is below to the exercise price, the contract is worthless to us; on the other hand if $S_T > K$ we can exercise our option (i.e., buy one share at the preassigned price K) and then sell the share immediately in the market for S_T.

In the last case the gain from this operation will be equal to $S_T - K$.

All combined, we can say that the buyer's gain is $f_T = (S_T - K)^+$.

Of course, one must pay certain *premium* C_T for the acquisition of this financial instrument, so that the *net profit of the buyer of the call option* is $f_T - C_T$.

The buyer purchasing the option can simply wait till the maturity date T, watching the dynamics of the pricess $(S_t)_{0 \le t \le T}$.

The position of the option writer is much more complicated because he must bear in mind his obligation to meet the terms of the contract, which requires him to not merely contemplate the changes in the pricess, but to use all financial means available to him to build a portfolio of securities

that ensures the final payment f_T.
If an option can be shown at an arbitrary (random!) instant $\tau \leq T$, then we call it an option of *American type.*
In practice, most options are American; this gives the buyer more freedom, allowing him to choose the exercise time τ.
In some sene the two types of options are equivalent in certain sense (the optimal exercise time τ of an American option is equal to T).
There exist several types of other options: *exotic, asian*, etc.

Central questions

1. What is the 'fair" price C_T (*option pricing*)
2. What the seller do to carry out the contract (*hedging problem*).

Mathamatical framework

Consider a financial market with $d+1$ *assets:*
one is riskless (*bond, bank account, money market account*)
d are risky (*stoks*)
being subject to random perturbations, so they have a high degree of uncertaintly
The assets are traded continuuously in a period of time T.

Let $(W_t)_{t\in[0,T]} = \left\{\left(W_t^{(1)}, ..., W_t^{(m)}\right)\right\}_{t\in[0,T]}$ be a m-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ (represent the noise or the random perturbation)
We define $\mathcal{F}_t$ as the σ-algebra generated by $(W_s)_{s\leq t}$ and we assume that $\mathcal{F}_T = \mathcal{F}$.

Definition 1.1. A financial market with $d+1$ assets is a system

$$\left\{(B_t)_{t\in[0,T]}, \left(S_t^{(1)}\right)_{t\in[0,T]}, ..., \left(S_t^{(d)}\right)_{t\in[0,T]}\right\},$$

where:
(a) $(B_t)_{t\in[0,T]}$is a random process (the bond), $B_t : (\Omega, \mathcal{F}, P) \longrightarrow R$, with $B_0 = 1$ for simplicity, and

$$B_t = 1 + \int_0^t r(s)B_s ds, t \in [0,T], \tag{1}$$

with $\{r(t)\}_{t\in[0,T]}$ a random process (*interest rate process*), so that

$$B_t = \exp\left\{\int_0^t r(s)ds\right\}.$$

(b) $(S_t)_{t\in T} = \left\{\left(S_t^{(1)}\right)_{t\in T}, ..., \left(S_t^{(d)}\right)_{t\in T}\right\}$ is a nonnegative d-dimensional process of the *prices* of risky assets given by

$$dS_t = g(t)dt + G(t)dW(t),$$

i.e.,

$$S_t = s_0 + \int_0^t g(s)ds + \int_0^t G(s)dW_s, \tag{2}$$

where:
(a) $s_0 = (s_1, ..., s_d)$ is a vector from R^d(*initial prices*) and g, G are some stochastic processes of corrresponding dimensions.
Componentwise (2) writes as

$$dS_t^{(i)} = g_i(t)dt + \sum_{j=1}^{m} g_{i,j}(t)dW_t^{(j)}, 1 \leq i \leq d. \tag{3}$$

The process

$$\tilde{S}_t^{(i)} = S_t^{(i)} \exp\left\{-\int_0^t r(s)ds\right\},$$

is known as the *discounted value of the i-th stock.*
Therefore the random vector $\left(B_t, S_t^{(1)}, \ldots, S_t^{(d)}\right)$ represents the *prices* of all $d+1$-*assets at time* t.
Depending of the form of interest rates of the pricess or the assets we obtain examples of known financial markets.

In the case of c*ontinuous liniar model*

$$dS_t = \mu(t)S_t dt + \sigma(t)S_t dW(t),$$

$\mu(t)$ is the *mean rate or return process* (represents the expected evolution of the stocks prices),
$\sigma(t)$ is known as the *volatility process.*

Black-Scholes: r, μ, σ *are deterministic*

Definition 1.2. (a) A *financial strategy (portfolio)* is a system $\left(f_t, f_t^{(1)}, ..., f_t^{(d)}\right)_{t\in[0,T]}$, with $f_t, f_t^{(i)}$ measurable and $\mathcal{F}_t$-adapted processes.
(b) *The value of portfolio at time* t *correspnding to* π is

$$V_t^\pi = f_t B_t + \sum_{i=1}^d f_t^{(i)} S_t^{(i)},\ t \in [0, T]\,, \tag{4}$$

the *discounted value* is

$$\tilde{V}_t^\pi = f_t + \sum_{i=1}^d f_t^{(i)} \tilde{S}_t^{(i)},\ t \in [0, T]\,.$$

A natural condition we should impose on the strategy is the *self-financing* condition,

$$dV_t^\pi = f_t dB_t + \sum_{i=1}^d f_t^{(i)} dS_t^{(i)},$$

or equivalently,

$$d\tilde{V}_t^\pi = \sum_{i=1}^d f_t^{(i)} d\tilde{S}_t^{(i)}, t \in [0, T]\,. \tag{5}$$

The self-financing property means that the stocks are traded at some discrete random times, so between the instants of tradings the variation of the portfolio value is due only to the changes in the assets prices; there are no withdraws of money and only the money gained by following the strategy can be reinvested.

If $\alpha \geq 0$, we say that the self-financing strategy π is $\alpha-$*admisible* if $V_t^\pi \geq -\alpha$ a.s. for alll t this means that it must be a limit when borrowing money that creditors accept.
Denote Π_α the $\alpha-$admisible strategies and $\Pi_+ = \bigcup_{\alpha \geq 0} \Pi_\alpha$.
Arbitrage: $\pi \in \Pi_+$ such that $V_0^\pi = 0$, $V_t^\pi \geq 0$ and $P(V_T^\pi > 0) > 0$ (*free lunch with vanishing risk).*
We interpret the arbitrage as a strategy which allows someone who doesn't invest anything to gain something, with a positive probability.
If there is not arbitrage the financial market is called *viable* (fair, rational market, no riskless profits).
Contigent claim (*derivative security*)*:* any nonnegatibe bounded random variable which is $\mathcal{F}_T$-measurable ($h = (S_T - K)^+$ is the case of Black-Scholes model).
Complet financial market: if for any contigent claim h there is a $\pi \in \Pi_+$ such that $V_T^\pi = h$ (every derivative security can be *hedged*).

Comment: *A very surprising and conforting fact is that there exists a interplay between the above economic concepts and the theory of martingales.*

Martingale measure (*risk neutral probability measure*)*:* any probability measure $\tilde{P}$ which is equivalent with P and for which the discounted pricess are martingales.
The complete answer to the two central questions:
1. What is the 'fair" price C_T (*option pricing*)
2. What the seller do to carry out the contract (*hedging problem*).

Theorem (*first fundamental theorem of asset pricing*). *A financial market is viable if and only if there exist a martingale measure.*

Theorem (*second fundamental theorem of asset pricing*). *A financial market is complete if and only if there exist a unique martingale measure.*

Corollary. *The liniar financial market (with* r, μ, σ *adapted and bounded processee) is viable and complete.*
In particular the Black-Scholes model is viable and complete.

Let $x \geq 0$ (inicial capital), h contigent claim.
A (x, h)-*hedging strategy*: $\pi \in \Pi_+$ with

$$V_0^\pi = x, V_a^\pi \geq h.$$

Denote by $SA(x, h)$ the family of all (x, h)-*hedging strategies*.

We define the *fair price* of the option h as

$$C(h) = \inf \{x \geq 0 : SA(x, h) \neq \emptyset\}. \tag{6}$$

Remark. It is clear that the above formula realizes the idea of satisfying both the seller (he can attain the claim) and also the buyer (in a certain sense he pays a minimal premium to the seller).

Theorem 1.3. *Under "smooth" conditions*

$$C(h) = E_{\tilde{P}}\left(he^{-\int_0^T r(s)ds}\right), \tag{7}$$

and there exists a strategy $\pi_0 \in \Pi_0$ *such that*

$$V_0^{\pi_0} = C(h), V_T^{\pi_0} = h \tag{8}$$

Theorem 1.4. *If the contigent claim* h *has the form* $h = f(S_T)$, *with* $f : R \longrightarrow R_+$ *measurable and* $E_{\tilde{P}}(h) = E_P(hZ_a) < \infty$. *then*

$$V_t^{\pi_0} = F(t, S_t), \tag{9}$$

$$F(t,x) = e^{-\int_t^T r(u)du} E_{\tilde{P}}\left[f\left(xe^{\int_t^T \sigma(u)d\tilde{W}_u + \int_t^T \left(r(u) - \frac{\sigma^2(u)}{2}\right)du}\right)\right], t \in [0,T], x \in R, \tag{10}$$

and the fair price is

$$C(h) = F(0, s_0). \tag{11}$$

Theorem 1.5 (*Formula Black-Scholes*). *For a call option* $f(x) = (x-K)^+$,

$$F(t,x) = \Phi(\alpha - a_1)x - Ke^{-\int_t^T r(u)du}\Phi(-a_1), \tag{12}$$

$$\Phi(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy, \alpha = \sqrt{\int_t^T \sigma^2(u)du},$$

$$\beta = e^{-\int_t^T r(u)du}, a_1 = \frac{\ln\frac{K\beta}{x}}{\alpha} + \frac{\alpha}{2}.$$

In particular

$$C\left((S_T - K)^+\right) = \Phi(\alpha - a_1)s_0 - Ke^{-\int_0^T r(u)du}\Phi(-a_1). \tag{13}$$

Remark. An important reason for which the Black-Merton-Scholes model was succeffuly is due to the fact that the results depend on only one parameter which is not directly observable, which is the volatility σ (for its estimation one can apply statistical methods).

2 Motivation from finance

Recall that the clasical Black-Scholes model has the dynamics the price of risky asset of the form

$$dS_t = S_t \left(\mu dt + \sigma dW_t \right), t \in [0, T], \tag{14}$$

where W is a standard Bm (the randomness of the stock price is due to W).
Its integral form is given by the lineat Itô equation

$$S_t = s_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dW_s, \tag{15}$$

with the explicit solution is given by

$$S_t = s_0 \exp \left\{ \mu t + \sigma W_t - \frac{\sigma^2}{2} t \right\}. \tag{16}$$

The second integral in (15) is the classical Itô integral.
Traditionally one assumes tthat there are no dividents, no transaction costs, same interest rate r for lending and saving on the bond.

Problems. The Black-Scholes pricing model is very satisfactory from the theoretical point of view.
Claims can be priced fairly and (in principle) one can even colculate the corresponding hedging portfolios and there are no arbitrage opportunities.

However, there is a problem with this model.

Let $0 = t_0 < t_1 < .... < t_n = T$ be a partition.
The model stipulates that the *log-returns*

$$\log \frac{S_{t_k}}{S_{t_{k-1}}} = \left(\mu - \frac{\sigma^2}{2} \right) (t_k - t_{k-1}) + \sigma \left(W_{t_k} - W_{t_{k-1}} \right),$$

are independent stationary normal random variables (due to the similar properties of W).

The are empirical studies (of financial time series) indicating that the log-returns are not independent neither stationary (even they are not Gaussian: we ignote in what follows this fact).
To overcome the first critical poins (non-independence, but preserving stationarity) of the log-returns it has been proposed that one should replace the Bm by a *fractional Brownian motion* (fBm for short) which captures the long-range dependence of the randomness. The first one to suggest this was Mandelbrot in the late sixties.
For the second critical point (nonindependence and nonstationarity) we propose the so called *sub-fractional Brownian motion* (sfBm).

Both above mentioned processes are not semimartingales and therefore the classical stochastic calculus does not apply.
A new approach to define the stochastic integral in (15) is necessary.

3 Processes with long-range dependence: fractional and sub-fractional Brownian motions

A stochastic process $(X_t)_{t\geq 0}$ exhibit *long-range dependence* if, for every $0 \leq u < v \leq s < t$,

$$cov(X_u - X_v, X_{s+\tau} - X_{t+\tau}) \sim r(u, v, s, t)\tau^{-\alpha} \text{ as } \tau \to \infty,$$

for some function $r(u, v, s, t)$ and $\alpha \in (0, 1)$, that is the dependence between $X_u - X_v$ and $X_{s+\tau} - X_{t+\tau}$ decays slowly as $\tau \to \infty$.
Another property we need is self-similaruty.
The centered process $(X_t)_{t\geq 0}$ is $\alpha-$*self-similar* if $(X_t)_{t\geq 0}$ and $(a^{-\alpha} X_t)_{t\geq 0}$ have the same distribution for any $a > 0$.

3.1 General properties

The *fractional Brownian motion* (fBm for short) is the best known and most used process with long-dependence property for models in telecommunication, turbulence, finance, etc. This process was first introduced by Kolmogorov (1940) and later studied by Mandelbrot and his coworkers (1968). The fBm is a continuous centered Gaussian process $\left(B_t^k\right)_{t\in R}$, starting from zero, with covariance

$$C_{B^k}(s,t) := E\left(B_t^k B_s^k\right) = \frac{1}{2}\left(|s|^{2k+1} + |t|^{2k+1} - |t-s|^{2k+1}\right), s,t \in R, \tag{17}$$

where $k \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ ($H = k + \frac{1}{2}$ is called Hurst parameter). The case $k = 0$ corresponds to the Brownian motion.

The *self-similarity* (with $\alpha = H$) *and stationarity of the increments* are two main properties for which fBm enjoyed success as a modeling tool. The fBm is the only continuous Gaussian process which is self-similar and has stationary increments.

An extension of Bm which preserves many properties of the fBm, but not the stationarity of the increments, is so called *sub-fractional Brownian motion* (*sfBm* for short), i.e.,a continuous Gaussian process $\left(S_t^k\right)_{t\geq 0}$, starting from zero, with covariance

$$C_{S^k}(s,t) := E\left(S_t^k S_s^k\right) = s^{2k+1} + t^{2k+1} - \frac{1}{2}\left[(s+t)^{2k+1} + |t-s|^{2k+1}\right], s,t \geq 0. \tag{18}$$

Next we assume $k \neq 0$.

The sfBm has properties analogous to those of fBm (see *Bojdecki-Gorostiza-Talarczyk: SPL-2004, CT-Stochastics-2007*):

(i_1) *Self-similarity*: For each $a > 0$ the processes $\left(S_{at}^k\right)_{t\geq 0}$ has the same distribution as $\left(a^{k+\frac{1}{2}} S_t^k\right)_{t\geq 0}$.

(i_2) *Covariance:* For all $s, t \geq 0$,

$$C_{S^k}(s,t) > 0,$$

$$C_{S^k}(s,t) > C_{B^k}(s,t) \text{ if } k \in \left(-\frac{1}{2}, 0\right),$$
$$C_{S^k}(s,t) < C_{B^k}(s,t) \text{ if } k \in \left(0, \frac{1}{2}\right).$$

(i_3) *Non-stationarity of increments:* For all $s \leq t$,

$$E\left[\left|S_t^k - S_s^k\right|^2\right] = -2^{2k}(t^{2k+1} + s^{2k+1}) + (t+s)^{2k+1} + (t-s)^{2k+1},$$
$$E\left(\left|S_t^k\right|^2\right) = (2 - 2^{2k})t^{2k+1},$$
$$(t-s)^{2k+1} \leq E\left[\left|S_t^k - S_s^k\right|^2\right] \leq (2 - 2^{2k})(t-s)^{2k+1} \text{ if } k \in \left(-\frac{1}{2}, 0\right),$$
$$(2 - 2^{2k})(t-s)^{2k+1} \leq E\left[\left|S_t^k - S_s^k\right|^2\right] \leq (t-s)^{2k+1} \text{ if } k \in \left(0, \frac{1}{2}\right).$$

(i_4) *Correlation of increments and long-range dependence:* For $0 \leq u < v \leq s < t$,define

$$R_{u,v,s,t}^k = E\left[\left(B_v^k - B_u^k\right)\left(B_t^k - B_s^k\right)\right],$$
$$C_{u,v,s,t}^k = E\left[\left(S_v^k - S_u^k\right)\left(S_t^k - S_s^k\right)\right].$$

Then

$$C_{u,v,s,t}^k = \frac{1}{2}\left[(t+u)^{2k+1} + (t-u)^{2k+1} + (s+v)^{2k+1} + (s-v)^{2k+1}\right.$$
$$\left. - (t+v)^{2k+1} - (t-v)^{2k+1} - (s+u)^{2k+1} - (s-u)^{2k+1}\right],$$
$$R_{u,v,s,t}^k < C_{u,v,s,t}^k < 0 \text{ if } k \in \left(-\frac{1}{2}, 0\right),$$
$$0 < C_{u,v,s,t}^k < R_{u,v,s,t}^k \text{ if } k \in \left(0, \frac{1}{2}\right).$$

For $u \geq 0, r > 0$ let $\rho_{B^k}(u,r)$ and $\rho_{S^k}(u,r)$ denote the correlation coefficients of $B_{u+r}^k - B_u^k,\ B_{u+2r}^k - B_{u+r}^k$ and $S_{u+r}^k - S_u^k,\ S_{u+2r}^k - S_{u+r}^k$. Then

$$\left|\rho_{S^k}(u,r)\right| \leq \left|\rho_{B^k}(u,r)\right|.$$

and we have the *long-range dependence*

$$R^k_{u,v,s+\tau,t+\tau} \sim k(2k+1)\,(t-s)\,(v-u)\,\tau^{2k-1} \text{ as } \tau \to \infty,$$

$$C^k_{u,v,s+\tau,t+\tau} \sim k(2k+1)(1-2k)\left(v^2-u^2\right)\tau^{2(k-1)} \text{ as } \tau \to \infty.$$

Therefore the covariance of increments of sfBm over non-overlapping intervals have the same sign but are smaller in absolute value than those of fBm and the increments on the intervals $[u, u+r]$, $[u+r, u+2r]$ are more weakly correlated than those of fBm. Moreover the long-range dependence decays at a higher rate for sfBm than for fBm (these properties justifies the name sfBm).

(i_5) *Short memory:* For each $a > 0$,

$$\sum_{n \geq a+1} cov\left(S^k_{a+1} - S^k_a, S^k_{n+1} - S^k_n\right) < \infty.$$

The above mentioned properties make sfBm a possible candidate for models which involve long-dependence, self-similarity and non-stationarity of the increments.

(i_6) S^k *is not a Markov process.*

(i_7) *Hölder paths:* For each $\varepsilon < k + \frac{1}{2}$ and each $T > 0$ there exists a random variable $K_{\varepsilon,T}$ such that

$$\left|S^k_t - S^k_s\right| \leq K_{\varepsilon,T}\left|t-s\right|^{k+\frac{1}{2}-\varepsilon},\ s,t \in [0,T],\ a.s.$$

(i_8) *Variation:* For each $T > 0$,

$$\sum_{i=0}^{n-1}\left|S^k_{\frac{(i+1)T}{n}} - S^k_{\frac{iT}{n}}\right|^p \overset{n\to\infty}{\longrightarrow} 0 \text{ if } p > \frac{2}{2k+1},\ \text{in } L^2$$

$$\sum_{i=0}^{n-1}\left|S^k_{\frac{(i+1)T}{n}} - S^k_{\frac{iT}{n}}\right|^p \overset{n\to\infty}{\longrightarrow} \rho_{\frac{2}{2k+1}}T \text{ if } p = \frac{2}{2k+1},\ \text{in } L^2$$

$$\sum_{i=0}^{n-1}\left|S^k_{\frac{(i+1)T}{n}} - S^k_{\frac{iT}{n}}\right|^p \overset{n\to\infty}{\longrightarrow} \infty \text{ if } p < \frac{2}{2k+1},\ \text{in } p$$

where $\rho_p = E\left(|N(0,1)|^p\right)$.

Such a result for fBm is obtained mainly by using self-similarity and stationarity of the increments (in particular ergodic theorem). For sfBm the lack of stationarity of the increments is replaced by linear regression.

(i_9) *If* W *is a Brownian motion independent of* S^k *and* $k > \frac{1}{4}$, *then the process* $S^k + W$ *is a semimartingale equivalent in law with* W.

(i_{10}) The *sfBm* S^k *is not a semimartingale.*

3.2 Pathwise integral with respect to sfBm

Since the fBm amd sfBm are not semimartingales, one cannot use the Itô theory to define stochastic integrals with respect to the..
However, one can define a pathwise, i.e. ω by ω, integrals as a refinement of the Riemann-Stieltjes integrals using $p-$variation.
For functions of one variable the Riemann-Stieltjes integral $\int_0^T f(t)dg(t)$ was extended to functions with unbounded variation, essentially by using fractional integrals or p-variation (see Dudley-Norvaisa-1998, 1999, Feyel-Pradelle-1999, Kondurar-1937, Mikosch-Norvaisa-2000, Young-1936 and Zähle-1998).
For a function $f : [0,T] \to R$ a partition $\pi : 0 = t_0 < ... < t_N = T$ and $p \geq 1$ we define the p-variation associated to π by

$$v_p(f,\pi) = \sum_i |f(t_i) - f(t_{i-1})|^p \, ,$$

Denote

$$v_p^0(f) = \lim_{|\pi_n| \to 0} v_p(f,\pi_n), \;\; v_p(f) = \sup_\pi v_p(f,\pi),$$

for all homogeneous partitions $\pi_n = (iT\delta_n)_i \, , \delta_n \longrightarrow 0$.
We say that f has *finite p-variation* if $v_p^0(f) < \infty$ and *bounded p-variation* if $v_p(f) < \infty$.

Remark. A function $f : [0,T] \longrightarrow R$ has bounded p-variation if and only if $f = g \circ h$, where $h : [0,T] \longrightarrow R$ is bounded nondecreasing nonnegative function and $g : [h(0), h(T)] \longrightarrow R$ is $\frac{1}{p}$-Hölder.

The family of functions with bounded p-variation is denoted by $\mathcal{W}_p$.
We denote by $H_{[0,T],\alpha}$ the class of all α-Hölder functions $f : [0,T] \longrightarrow R$ with $f(0) = 0$ and define

$$\|f\|_{[0,T],\alpha} = \sup_{u \neq v, 0 \leq u < v \leq T} \frac{|f(u) - f(v)|}{(v-u)^\alpha}.$$

Remark. (Young-1936, Dudley-Norvaisa-1999). If $f \in \mathcal{W}_p, g \in \mathcal{W}_q$, $\frac{1}{p} + \frac{1}{q} > 1$ and f, g have no common discontinuities, then the Stieltjes integral $\int_0^T f(t)dg(t)$ exists as limit of the corresponding Riemann-Stieltjes sums.
In particular if f is α-Hölder, g is β-Hölder with $\alpha + \beta > 1$, then the Stieltjes integral $\int_0^t f(s)dg(s)$ exists and is β-Hölder. Moreover for every $0 < \varepsilon < \alpha + \beta - 1$,

$$\left| \int_0^T f(s)dg(s) \right| \leq C(\alpha, \beta) \left\| f \right\|_{[0,T],\alpha} \left\| g \right\|_{[0,T],\beta} T^{1+\varepsilon}, \tag{19}$$

(see Feyel-Pradelle-1999).
Concerning the variation of sfBm we have the following result.
Proposition 3.1

$$v_p^0(S^k) = 0,\ v_p(S^k) < \infty \ \mathit{if}\, p > \frac{2}{2k+1}, \tag{20}$$

$$v^0_{\frac{2}{2k+1}}(S^k) = v_{\frac{2}{2k+1}}(S^k) = \rho_{\frac{2}{2k+1}} T, \tag{21}$$

$$v_p^0(S^k) = v_p(S^k) = \infty \ \mathit{if}\, p < \frac{2}{2k+1}. \tag{22}$$

Remark. From the above proposition it follows that a.s. $S^k \in \mathcal{W}_p$ if and only if $p \geq \frac{2}{2k+1}$.
Moreover, for every process $(u_t)_{t\in[0,T]}$ with paths a.s. in $\mathcal{W}_q$ with $q < \frac{2}{1-2k}$, the Riemann-Stieltjes integral $\int_0^t u_r dS_r^k$ is well defined a.s. In particular if u has α-Hölder paths for some $\alpha > \frac{1-2k}{2}$, then the Riemann-Stieltjes integral $\int_0^t u_r dS_r^k$ is well defined and has β–Hölder paths, for every $\beta < k + \frac{1}{2}$.

Since every Riemann-Stieltjes integral obeys the change of variable formula, we have the following result.

Theorem 3.2. *If $F(t,x) \in C^1$ and the mapping $t \longrightarrow \frac{\partial F}{\partial x}(t, S_t^k) \in W_q$ with $q < \frac{2}{1-2k}$, then for all $s, t \in [0,T]$,*

$$F(t, S_t^k) - F(s, S_s^k) = \int_s^t \frac{\partial F}{\partial x}(r, S_r^k) dS_r^k + \int_s^t \frac{\partial F}{\partial t}(r, S_r^k) dr. \tag{23}$$

Remark. (a) The pathwise integral *may not exist* : for example for $k \in \left(-\frac{1}{2}, 0\right)$ the integral $\int_0^T S_t^k dS_t^k$ does not exists. Indeed if we assume that the integral exists, then using (22) we obtain

$$\infty = v_2^0(S^k) = \lim_{n\to\infty} \sum_i \left| S^k_{\frac{iT}{n}} - S^k_{\frac{(i-1)T}{n}} \right|^2 = \lim_{n\to\infty} \sum_i S^k_{\frac{iT}{n}} \left(S^k_{\frac{iT}{n}} - S^k_{\frac{(i-1)T}{n}} \right)$$

$$- \lim_{n\to\infty} \sum_i S^k_{\frac{(i-1)T}{n}} \left(S^k_{\frac{iT}{n}} - S^k_{\frac{(i-1)T}{n}} \right) = \int_0^T S_t^k dS_t^k - \int_0^T S_t^k dS_t^k = 0.$$

(b) If the pathwise integral exists, then it *may have not zero expectation*: for example if $k \in \left(0, \frac{1}{2}\right)$, by using (23) we obtain

$$E\left(\int_0^T S_t^k dS_t^k\right) = \frac{1}{2} E\left(\left|S_T^k\right|^2\right) = (1 - 2^{2k-1}) T^{2k+1}.$$

Remark. *We have similar results for fBm.*

4 Anticipating calculus

4.1 Multiple Wiener-Itô integrals

Let $(X_t)_{t\geq 0}$ be a real valued centered Gaussian process defined on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{E}$ be the family of elementary deterministic functions, i.e., of functions of the form $f = \Sigma_{j=1}^{n-1} f_j 1_{[t_j, t_{j+1})}$, $f_j \in R$ and $t_0 = 0 < t_1 < ... < t_n$. *We assume that* $\mathcal{F} = \mathcal{B}(X_t : t \in T)$.
For $f \in \mathcal{E}$ we define

$$I(f) = \sum_{j=1}^{n-1} f_j \left(X_{t_{j+1}} - X_{t_j} \right), \tag{24}$$

and the symmetric bilinear form on $\mathcal{E}$,

$$\langle f, g \rangle_X = E\left(I(f)I(g)\right).$$

The closure of $\mathcal{E}$ with respect to the above inner product is denoted by Λ_X and we call it *the domain of the Wiener integral.* For $f \in \Lambda_X$ we denote by $\int_T f(t)dX_t$ or $X(f)$ the extension by continuity of (24) to Λ_X and we call it *the Wiener integral* of f with respect to X.
The description of the Hilbert space Λ_X and the explicit expression of $X(f)$ is not easy in the most cases.
In the case of Bm $\Lambda_X = L^2(R_+)$ and $\int_T f(t)dX_t$ is the standard Wiener integral.
The process $\{X(f)\}_{f\in\Lambda_X}$ is a centered Gaussian process, the so called an *isonormal process.*
For an integer $q \geq 1$ we denote by $\Lambda_X^{\odot q}$ the symmetric tensor product equipped with the modified norm $\sqrt{q!}\left\|.\right\|_{\Lambda_X^{\odot q}}$. In some cases $\Lambda_X = L^2(A, \mathcal{A}, \mu)$, where μ is a σ-finite and non-atomic measure and in this context $\Lambda_X^{\odot q}$ can be identified with $L_s^2(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$, the space of symmetric square integrable functions on A^q.
Next, for any unexplained concept or result on Malliavin calculus, the reader is referred to Nualart (2006).

For every $q \geq 1$, we write $\mathcal{H}_q$ for the qth Wiener chaos, which is the closed subsace of $L^2(\Omega, \mathcal{F}, P)$ generated by $H_q(X(h))$, $h \in \Lambda_X$, $\|h\|_{\Lambda_X} = 1$, where H_q is the Hermite polynomial of order q , defined as

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}}, x \in R, q \geq 1,$$

For any $q \geq 1$, the mapping

$$I_q(h^{\otimes q}) = q! H_q(X(h)),$$

can be extended to a linear isometry between $\Lambda_X^{\odot q}$ equipped with the modified norm $\sqrt{q!}\,\|\|_{\Lambda_X^{\odot q}}$ and $\mathcal{H}_q$.
For $f \in \Lambda_X^{\odot q}$ we denote by $I_q(f)$ the multiple stochastic integral of f with respect to X obtained by the above isometry.
Recall that in the case $\Lambda_X = L^2(A, \mathcal{A}, \mu)$, the random variable $I_q(f)$ agrees with the multiple Wiener-Itô integral (by convention $\mathcal{H}_0 = R$).
For $q = 0$ we put $I_0(c) = c \in R$.
If $(e_k)_{k\geq 1}$ is a CONS in Λ_X and $f \in \Lambda_X^{\odot q}$, $g \in \Lambda_X^{\odot q}$, then for any $r = 0, ..., p \wedge q$, we define the rth contraction $f \otimes_r g$ as the element of $\Lambda_X^{\odot(p+q-2r)}$ defined as

$$f \otimes_r g = \sum_{i_1,...,i_r=1}^{\infty} \langle f, e_{i_1} \otimes ... \otimes e_{i_r} \rangle_{\Lambda_X^{\otimes r}} \langle g, e_{i_1} \otimes ... \otimes e_{i_r} \rangle_{\Lambda_X^{\odot r}}.$$

In the special case where
$\Lambda_X = L^2(A, \mathcal{A}, \mu)$, one has for $1 \leq r \leq p \wedge q$,
$f \otimes_r g =$

$$\int_{A^r} f(t_1, ..., t_{p-r}, s_1, ..., s_r) g(t_{p-r+1}, ..., t_{p+q-2r}, s_1, ..., s_r) d\mu(s_1)...d\mu(s_r),$$

and $f \otimes_0 g = f \otimes g$.
The following product (multiplication) formula is very useful: if $f \in \Lambda_X^{\odot p}$, $g \in \Lambda_X^{\odot q}$, then

$$I_p(f) I_q(g) = \sum_{r=0}^{p\wedge q} r! C_p^r C_q^r I_{p+q-2r}(f \otimes_r g). \tag{25}$$

Chaotic decomposition. It is well known that the space $L^2(\Omega, \mathcal{F}, P)$ is the orthogonal su of the spaces $\mathcal{H}_q$, i.e., every $Z \in L^2(\Omega, \mathcal{F}, P)$ admits the chaos expansion

$$Z = \sum_{q=0}^{\infty} I_q(f_q), \tag{26}$$

where $f_0 = E(Z)$ and the kernels $(f_q)_{q\geq 1}$, $f_q \in \Lambda_X^{\odot q}$ are uniquely determined.

4.2 Basic of Malliavin calculus

Define

$$\mathcal{S} : Z = g\left(X(h_1), ..., X(h_n)\right), g \in C_c^{\infty}, h_i \in \Lambda_X.$$

Malliavin derivative:

$$DZ = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\left(X(h_1), ..., X(h_n)\right) h_i,$$

and by iteration $D^q Z$ as an element of $L^2(\Omega, \Lambda_X^{\odot q})$. For $q, p \geq 1$, $D^{q,p}$ denotes the closure of $\mathcal{S}$ with respect to the norm

$$\|Z\|_{q,p}^p = E\left(|Z|^p\right) + \sum_{i=1}^{q} E\left(\left\|D^i Z\right\|_{\Lambda_X^{\odot i}}^p\right).$$

If Z is as in (26), then

$$Z \in D^{1,2} \Leftrightarrow E\left(\|DZ\|_{\Lambda_X}^2\right) = \sum_{q=1}^{\infty} q \left\|I_q(f_q)\right\|_{L^2(\Omega)}^2 < \infty,$$

and in particular, if $\Lambda_X = L^2\left(A, \mathcal{A}, \mu\right)$,

$$D_t Z = \sum_{q=1}^{\infty} q I_{q-1}(f_q(., t)) \in L^2\left(A \times \Omega\right).$$

Chain rule: If $g \in C_b^1$ or g is a polynomial in d variables and $Z = \left(Z_1, ..., Z_d\right)$, $Z_i \in D^{1,2}$, then

$$Dg(Z) = \sum_{i=1}^{d} \frac{\partial g}{\partial x_i}(Z) DZ_i.$$

Skorohod integral (*anticipating integral, divergence operator*)*:* δ is the adjoint of D :

$$u \in L^2(\Omega, \Lambda_X) \in Dom(\delta) \Leftrightarrow \left|E\left(\langle DZ, u\rangle_{\Lambda_X}\right)\right| \leq c_u \|Z\|_{L^2(\Omega)}, \forall Z \in D^{1,2}.$$

Duality relation (*integration by parts formula*):

$$E\left(Z\delta(u)\right) = E\left(\langle DZ, u\rangle_{\Lambda_X}\right), \forall\ Z \in D^{1,2}.$$

4.3 The case of sub-fractional Brownian motion

Let $f : [0,T] \longrightarrow R$ be a measurable application and $\alpha \in R, \sigma, \eta \in R$. We define the *Erdély-Kober-type fractional integral*

$$\left(I^{\alpha}_{T-,\sigma,\eta}f\right)(s) = \frac{\sigma s^{\sigma\eta}}{\Gamma(\alpha)}\int_s^T \frac{t^{\sigma(1-\alpha-\eta)-1}f(t)}{\left(t^{\sigma}-s^{\sigma}\right)^{1-\alpha}}dt,\ s \in [0,T],\ \alpha > 0, \tag{27}$$

$$\left(I^{\alpha}_{T-,\sigma,\eta}f\right)(s) = s^{\sigma\eta}\left(-\frac{d}{\sigma s^{\sigma-1}ds}\right)^n s^{\sigma(n-\eta)}\left(I^{\alpha+n}_{T-,\sigma,\eta-n}f\right)(s),\ s \in [0,T],\ \alpha > -n, \tag{28}$$

$$\left(I^{\alpha}_{0+,\sigma,\eta}f\right)(s) = \frac{\sigma s^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_0^s \frac{t^{\sigma\eta+\sigma-1}f(t)}{\left(s^{\sigma}-t^{\sigma}\right)^{1-\alpha}}dt,\ s \in [0,T],\ \alpha > 0, \tag{29}$$

$$\left(I^{\alpha}_{0+,\sigma,\eta}f\right)(s) = s^{-\sigma(\alpha+\eta)}\left(\frac{d}{\sigma s^{\sigma-1}ds}\right)^n s^{\sigma(\alpha+n+\eta)}\left(I^{\alpha+n}_{0+,\sigma,\eta}f\right)(s),\ s \in [0,T],\ \alpha > -n. \tag{30}$$

We introduce the following kernels

$$n(t,s) = \frac{\sqrt{\pi}}{2^k} I^k_{T-,2,\frac{1-k}{2}}\left(u^k 1_{[0,t)}\right)(s), \tag{31}$$

$$\psi(t,s) = \frac{s^k}{\Gamma(1-k)}\left[t^{k-1}\left(t^2-s^2\right)^{-k} - (k-1)\int_s^t \left(u^2-s^2\right)^{-k} u^{k-1}du\right] 1_{(0,t)}(s). \tag{32}$$

Denote by $\Lambda^{sf}_{k,T}$ the domain of the Wiener integral for sub-fBm.
Remark. The function $\psi(t,.) \in \Lambda^{sf}_{k,T}$ and satisfies (uniquely) the equality

$$\frac{\sqrt{\pi}}{2^k} I^k_{T-,2,\frac{1-k}{2}}\left(u^k \psi(t,.)\right)(s) = 1_{(0,t)}. \tag{33}$$

Theorem 4.1 (Dzhaparidze-Van Zanten-2004, CT-2009). *The process*

$$W_t^k = \int_0^t \psi(t,s) dS_s^k,$$

is the unique Brownian motion such that

$$S_t^k = c_k \int_0^t n(t,s) dW_s^k, t \in [0,T], \tag{34}$$

$$c_k^2 = \frac{\Gamma(2k+2) \sin \pi \left(k+\frac{1}{2}\right)}{\pi}. \tag{35}$$

Moreover S^k and W^k generate the same filtration.

Theorem 4.2 (CT-2009)**.** (*i*) *If $-\frac{1}{2} < k < 0$, then the space $\left(\Lambda_{k,T}^{sf}, \langle .,. \rangle_{\Lambda_{k,T}^{sf}}\right)$, where*

$$\Lambda_{k,T}^{sf} = \left\{ f : [0,T] \to R : \exists \varphi_f \in L^2([0,T]), I_{T-,2,\frac{k+1}{2}}^{-k} \left(\frac{2^k}{\sqrt{\pi}} \varphi_f \right)(t) = t^k f(t) \right\}, \tag{36}$$

$$\langle f, g \rangle_{\Lambda_{k,T}^{sf}} = c_k^2 \int_0^T \varphi_f(t) \varphi_g(t) dt, \tag{37}$$

is the domain of the Wiener integral and

$$\int_0^T f(t) dS_t^k = c_k \int_0^T \varphi_f(t) dW_t^k. \tag{38}$$

(*ii*) *If $0 < k < \frac{1}{2}$, then the space $\left(\Lambda_{k,T}^{sf}, \langle .,. \rangle_{\Lambda_{k,T}^{sf}}\right)$, where*

$$\Lambda_{k,T}^{sf} = \left\{ f \in \mathcal{D}(0,T)' : \exists f^* \in \mathcal{S}', \ f^* \ odd, \ supp(f^*) \subset [-T,T] \right.$$
$$\left. f^* |_{[0,T]} = f, \ \int_R \left| \widehat{f^*}(x) \right|^2 |x|^{-2k} dx < \infty \right\}, \tag{39}$$

$$\langle f, g \rangle_{\Lambda_{k,T}^{sf}} = \frac{c_k^2}{2} \int_R \widehat{f^*}(x) \overline{\widehat{g^*}(x)} |x|^{-2k} dx, \tag{40}$$

is the domain of the Wiener integral.
If we define

$$|\Lambda|_{k,T}^{sf} = \left\{ f : [0,T] \longrightarrow R : I_{T-,2,\frac{1-k}{2}}^{k} \left(u^k |f| \right) \in L^2([0,T]) \right\}, \tag{41}$$

then we have the strict inclusion $|\Lambda|_{k,T}^{sf} \subset \Lambda_{k,T}^{sf}$ *and*

$$\int_0^T f(t)dS_t^k = c_k \int_0^T I_{T-,2,\frac{1-k}{2}}^k \left(\frac{\sqrt{\pi}}{2^k} u^k f \right)(t)dW_t^k, \ f \in L^2\left([0,T]\right). \tag{42}$$

Moreover, if $k \in \left(0, \frac{1}{2}\right), \ f \in |\Lambda|_{k,T}^{sf}$,

$$\begin{aligned} \langle f, g \rangle_{\Lambda_{k,T}^{sf}} &= c_k^2 \left\langle I_{T-,2,\frac{1-k}{2}}^k \left(\frac{\sqrt{\pi}}{2^k} u^k f \right), I_{T-,2,\frac{1-k}{2}}^k \left(\frac{\sqrt{\pi}}{2^k} u^k g \right) \right\rangle_{L^2([0,T])} \\ &= \int_0^T \int_0^T f(u)g(v)\varphi_k(u,v)dudv. \qquad (43) \\ \varphi_k(u,v) &= k(2k+1)\left[|u-v|^{2k-1} - (u+v)^{2k-1} \right]. \end{aligned}$$

Proposition 4.2. *The inclusion* $\Lambda_{k,T}^{f} \subset \Lambda_{k,T}^{sf}$ *holds.*

Theorem 4.4 (*Prediction*). *For every* $0 \le t \le T$,

$$\hat{S}_{T|t} := E\left[S_T^k \mid \mathcal{F}_t^{S^k}\right] = S_t^k + \int_0^t \Psi_{t,T}(u) dS_u^k$$
$$= c_k \int_0^t n(T,u) dW_u^k. \tag{44}$$

In particular (since $n(T,u) > 0$ *on* $(0,T)$*) we have the equality* $\mathcal{F}_t^{S^k} = \mathcal{F}_t^{\hat{S}^k}$.

Denote

$$d_k = \frac{2^k}{c_k \Gamma(1-k)\sqrt{\pi}},$$

The process

$$M_t^k = d_k \int_0^t s^{-k} dW_s^k, \tag{45}$$

is called the *sub-fractional fundamental martingale.*
The following result is straightforward.
Lemma. *For every* $0 \le s \le t \le T$,

$$cov\left(S_s^k, M_t^k\right) = s, \; E\left(\left|M_t^k\right|^2\right) = \frac{d_k^2}{1-2k} t^{1-2k}.$$

In particular $M_t^k - M_s^k$ *is independent of* $\mathcal{F}_s^{S^k}$ *and* $\mathcal{F}_s^{S^k} = \mathcal{F}_s^{M^k} = \mathcal{F}_s^{W^k}$.

For $f : [0,T] \to R$ with $\int_0^T f^2(s) s^{-2k} ds < \infty$ define the probability Q_f by

$$\frac{dQ_f}{dP} \mid_{\mathcal{F}_t^{S^k}} = \exp\left(\int_0^t f(s) dM_s^k - \frac{1}{2}\int_0^t f^2(s) \left\langle M^k \right\rangle_s\right)$$
$$= \exp\left(\int_0^t f(s) dM_s^k - \frac{d_k^2}{2}\int_0^t f^2(s) s^{-2k} ds\right), \tag{46}$$

and denote

$$\left(\Psi_k f\right)(s) = \frac{1}{\Gamma(1-k)} I_{0+,2,-k}^k f(s). \tag{47}$$

Theorem 4.5 (*Girsanov*). *For f as above, the process*

$$S_t^k - \int_0^t (\Psi_k f)(s)\, ds, t \in [0, T],$$

is a Q_f–sfBm.
In particular if $f \equiv a \in R$ it follows that the process $\left(S_t^k - at\right)_{t\in[0,T]}$ is Q_a–sfBm.

4.4 Multiple sub-fractional integrals and anticipating stochastic calculus for sfBm

Multiple integrals w.r.t. fBm were introduced by Desgupta-Kallianpur-PTRF-1999, and Duncan, Hu and Pasik-Duncan-SIAM J. Control Optim.-2000 for the fBm. The techniques used in these papers involve Wick product and reproducing kernel Hilbert space theory.

We study multiple fractional and subfractional integrals by using this *transfer principle* from multiple Brownian integrals via a Gamma type operator. Then the chaos form of the sub-fractional anticipating integral is considered.

We assume that $k \in \left(0, \frac{1}{2}\right)$.

For a function $f : [0,T]^n \to R$ we consider the $n-$dimensional form $I^{\alpha,n}_{T-,\sigma,\eta}f$ of the *Erdély-Kober-type fractional integrals* (27),

$$\left(I^{\alpha,n}_{T-,\sigma,\eta}f\right)(s_1,..,s_n) = \left(\frac{\sigma}{\Gamma(\alpha)}\right)^n \prod_{j=1}^n s_j^{\sigma\eta}$$

$$\times \int_{s_1}^T .. \int_{s_n}^T \prod_{j=1}^n \frac{t_j^{\sigma(1-\alpha-\eta)-1}}{\left(t_j^\sigma - s_j^\sigma\right)^{1-\alpha}} f(t_1,..,t_n)dt_1..dt_n,\ s_j \in [0,T],\ \alpha > 0. \quad (48)$$

$$\left(I^{\alpha,n}_{0+,\sigma,\eta}f\right)(s) = \left(\frac{\sigma}{\Gamma(\alpha)}\right)^n \prod_{j=1}^n s_j^{-\sigma(\alpha+\eta)}$$

$$\times \int_0^{s_1} .. \int_0^{s_n} \prod_{j=1}^n \frac{t_j^{\sigma\eta+\sigma-1} f((t_1,..,t_n)}{\left(s_j^\sigma - t_j^\sigma\right)^{1-\alpha}} dt_1..dt_n,\ s_j \in [0,T],\ \alpha > 0. \quad (49)$$

We shall denote by $\mathcal{I}^{0,k}_n(f)$ the multiple Wiener-Itô integral with respect to W^k and we introduce the space

$$|\Lambda^n|_{\Lambda^{sf}_{k,T}} = \left\{ f : [0,T]^n \to R : I^{k,n}_{T-,2,\frac{1-k}{2}}(|f|) \in L^2\left([0,T]^n\right)\right\}.$$

Definition. If $f \in |\Lambda^n|_{k,T}^{sf}$, f symmetric, then we define the *multiple sub-fractional integral of f with respect to S^k* by

$$\mathcal{I}_n^k(f) = \left(c_k \frac{\sqrt{\pi}}{2^k}\right)^n \mathcal{I}_n^{0,k}\left(I_{T-,2,\frac{1-k}{2}}^{k,n}\left(\prod_{j=1}^n u_j^k f\right)\right), \tag{50}$$

We have the equalities

$$\|f\|_{|\Lambda^n|_{k,T}^{sf}}^2 := \left\|\mathcal{I}_n^k(f)\right\|_{L^2(\Omega,\mathcal{F},P)}^2 = \left(c_k \frac{\sqrt{\pi}}{2^k}\right)^{2n} \left\|I_{T-,2,\frac{1-k}{2}}^{k,n}\left(\left(\prod_{j=1}^n u_j^k\right) f\right)\right\|_{L^2([0,T]^n)}^2$$

$$= \int_{[0,T]^{2n}} f(u_1,...,u_n) f(v_1,...,v_n) \prod_{j=1}^n \varphi_k(u_j,v_j) du_j dv_j. \tag{51}$$

Remark. Like in the fBm case the multiplication by an indicator function can increase the norm in $\Lambda_{k,T}^{sf}$.
Indeed, for $0 < a < T$ and $c \in \left[1, \frac{T}{a}\right]$ define

$$f_{c,a} = 1_{(0,a)} - 1_{(a,ac)}, \tag{52}$$

$$g(c) = 4 - 2^{2(k+1)} - (2 + 2^{2k}) c^{2k+1} + 2(c+1)^{2k+1} + 2(c-1)^{2k+1}.$$

Since

$$g'(1) = 2(2k+1)(2^{2k-1} - 2) < 0,$$

it follows that there exists $x_0 > 1$ such that the function g in decreasing on $[1, x_0]$.
Then there exist $a > 1$ and $1 < c_2^0 < c_1^0 < x_0$ such that

$$\|f_{c,a}\|_{\Lambda_{k,T}^{sf}} < 1, \ \forall c \in \left[c_1^0, x_0\right], \tag{53}$$

$$\|f_{c,a}\|_{\Lambda_{k,T}^{sf}} > 1, \ \forall c \in \left(1, c_2^0\right].$$

In particular

$$\left\|f_{c,a} 1_{(0,t)}\right\|_{\Lambda_{k,T}^{sf}} > 1, \ \forall t \in \left(a, ac_2^0\right]. \tag{54}$$

We can now define the space

$$\left|\mathcal{L}_k^2\right| = \left\{F \in L^2(\Omega,\mathcal{F},P) : F = \sum_{n=0}^{\infty} \mathcal{I}_n^k(f_n), \ f_n \in |\Lambda^n|_k^{sf}, \ f_n \text{ symmetric}\right\}.$$

Remark. Since $|\Lambda^n|_{k,T}^{sf}$ is not complete, $|\mathcal{L}_k^2|$ is a strict subspace of $L^2(\Omega, \mathcal{F}, P)$.
Following the ideas of [21] we introduce the following
Definition. For $F \in |\mathcal{L}_k^2|$ and $t \in [0,T]$, we define the *sub-fractional quasi-conditional expectation of F with respect to $\mathcal{F}_t^{S^k}$* by

$$\tilde{E}\left[F \mid \mathcal{F}_t^{S^k}\right] = \sum_{n=0}^{\infty} \mathcal{I}_n^k(1_{(0,t)^n} f_n), \tag{55}$$

provided the series converges in $L^2(\Omega, \mathcal{F}, P)$,i.e.,

$$\sum_{n=1}^{\infty} n! \left\| 1_{(0,t)^n} f_n \right\|_{|\Lambda^n|_{k,T}^{sf}}^2 < \infty. \tag{56}$$

Remark. To see the difference between conditional expectation and quasi-conditional expectation note that (see 44) for $t \in (0,T)$,

$$\tilde{E}\left[S_T^k \mid \mathcal{F}_t^{S^k}\right] = S_t^k,$$

$$E\left[S_T^k \mid \mathcal{F}_t^{S^k}\right] = S_t^k + \int_0^t \Psi_{t,T}(u) dS_u^k.$$

The sub-fractional quasi-conditional expectation need not exist as the next result shows.

Proposition 4.6. *Consider the sequence of kernels* $f_n = (n!)^{-\frac{1}{2}} f_{c,a}^{\otimes n}$, *where* $f_{c,a}$ *is given by* (52).
Then the following statements hold:
(a) *The random variable* $F = \sum_{n=1}^{\infty} \mathcal{I}_n^k(f_n) \in |\mathcal{L}_k^2|$.
(b) *For every* $t \in (a, ac_2^0]$, *with* a, c_2^0 *defined in Remark 4.2,* $\tilde{E}\left[F \mid \mathcal{F}_t^{S^k}\right]$ *is not well defined.*

Definition. The random variable $F = \sum_{n=1}^{\infty} \mathcal{I}_n^k(f_n) \in |\mathcal{L}_k^2|$ is *sub-fractional Malliavin differentiable* if

$$D_t^k F = \sum_{n=1}^{\infty} n \mathcal{I}_{n-1}^k \left(f_n(.,t)\right), \tag{57}$$

converges in $|\mathcal{L}_k^2|$ for a.e. $t \in [0,T]$, i.e.,

$$\sum_{n=1}^{\infty} nn! \left\| f_n(.,t) \right\|^2_{\left|\Lambda^{n-1}\right|_{k,T}^{sf}} < \infty. \tag{58}$$

The *sub-fractional Clark-Ocone derivative at time* t of $F \in |\mathcal{L}_k^2|$ is defined by

$$\nabla_t^k F = \tilde{E}\left[D_t^k F \mid \mathcal{F}_t^{S^k}\right], \tag{59}$$

provided F is sub-fractional Malliavin differentiable and the quasi-conditional expectation exists, i.e.,

$$\sum_{n=1}^{\infty} nn! \left\| 1_{(0,t)^{n-1}} f_n(.,t) \right\|^2_{\left|\Lambda^{n-1}\right|_{k,T}^{sf}} < \infty. \tag{60}$$

.
The sub-fractional Clark-Ocone derivative need not exist as the next result shows.

Proposition 4.7. *Consider the sequence of kernels* $f_n = (nn!)^{-\frac{1}{2}} f_{c,a}^{\otimes n}$, *where* $f_{c,a}$ *is given by* (52).
Then the following statements hold:
(a) *The random variable* $F = \Sigma_{n=0}^{\infty} \mathcal{I}_n^k(f_n)$ *is sub-fractional Malliavin differentiable and*

$$\int_{[0,T]^2} E\left(\left|D_s^k F\right|\left|D_t^k F\right|\right)\varphi_k(s,t)dsdt < \infty. \tag{61}$$

(b) *For every* $t \in (a, ac_2^0]$, *with* a, c_2^0 *defined in Remark 4.2,* $\tilde{E}\left[F \mid \mathcal{F}_t^{S^k}\right]$ *is not well defined.*

Consider the set $\left|\mathcal{S}^{ch}\right|_{k,T}^{sf}$ of all measurable processes $(u_t)_{t\in[0,T]}$ such that:
(a) $u_t \in |\mathcal{L}_k^2|$ for a.e. $t \in [0,T]$ and

$$u_t = \sum_{n=0}^{\infty} \mathcal{I}_n^k\left(f_n(.,t)\right), \tag{62}$$

with $f_n(.,t) \in |\Lambda^n|_{k,T}^{sf}$, $f_n \in |\Lambda^{n+1}|_{k,T}^{sf}$.
(b) The following series is convergent

$$\sum_{n=1}^{\infty} (n+1)! \left|\left|\left|sym\left(f_n\right)\right|\right|\right|_{|\Lambda^{n+1}|_{k,T}^{sf}}^2 < \infty.$$

Definition. For a process $u \in \left|\mathcal{S}^{ch}\right|_{k,T}^{sf}$ define the *chaos sub-fractional Skorohod integral of* u with respect to S^k by

$$\delta_{k,T}^{ch}(u) := \sum_{n=0}^{\infty} \mathcal{I}_{n+1}^k\left(sym\left(f_n\right)\right).$$

From (b) it follows that the previous converges in $|\mathcal{L}_k^2|$.

Define $\left|\mathcal{D}_k^{1,2}\right|$ as the family of all $F = \Sigma_{n=1}^{\infty} \mathcal{I}_n^k(f_n) \in |\mathcal{L}_k^2|$ such that

$$\sum_{n=1}^{\infty} nn! \left|\left|\left|f_n\right|\right|\right|_{|\Lambda^n|_{k,T}^{sf}}^2 < \infty. \tag{63}$$

Theorem 4.8 (*Clark-Ocone representation formula*). *If* $F \in \left|\mathcal{D}_k^{1,2}\right|$ *then the clark-Ocone derivative exists and satisfies*

$$\nabla_t^k F = \sum_{n=1}^{\infty} n\mathcal{I}_{n-1}^k\left(1_{(0,t)^{n-1}} f_n\left(.,t\right)\right), \tag{64}$$

$$\int_{[0,T]^2} E\left(\left|\nabla_s^k F\right|\left|\nabla_t^k F\right|\right)\varphi_k(s,t)dsdt < \infty. \tag{65}$$

Moreover, $\nabla^k F$ *is sub-fractional Skorohod integrable and the following Clark-Ocone representation formula holds*

$$F = E(F) + \delta_{k,T}^{ch}\left(\nabla^k F\right). \tag{66}$$

Remark. A representation of a random variable $F \in \left|\mathcal{L}_k^2\right|$ in the form

$$F = E(F) + \delta_{k,T}^{ch}\left(u\right), \tag{67}$$

need not be unique.
For example: By using the product formula it is easily see that we have the relations

$$\left(S_T^k\right)^2 = \delta_{k,T}^{ch}\left(S_T^k\right) + (2-2^{2k})T^{2k+1},$$
$$\left(S_T^k\right)^2 = \delta_{k,T}^{ch}\left(2S_.^k\right) + (2-2^{2k})T^{2k+1}.$$

Proposition 4.9. *A representation of the form* (67) *with u adapted is unique.*

Remark. In (66) the integrand is adapted.

Proposition 4.10. *For $0 < \tau < \sigma$ and $F \in L^2\left(R, N(0,\sigma^2)\right)$ define*

$$G_{\sigma,\tau}^F(x) = \frac{1}{\sqrt{2\pi\left(\sigma^2 - \tau^2\right)}} \int_R F(y) e^{-\frac{(x-y)}{2\left(\sigma^2-\tau^2\right)}^2} dy. \tag{68}$$

If $f \in |\Lambda|_{k,T}^{sf}$, $(a,b) \subset [0,T]$ and $\left\|1_{(a,b)} f\right\|_{\Lambda_{k,T}^{sf}} < \|f\|_{\Lambda_{k,T}^{sf}}$, then for every $F \in L^2\left(R, N(0, \|f\|_{\Lambda_{k,T}^{sf}}^2)\right)$ we have the equality

$$\tilde{E}\left[F\left(\mathcal{I}_1^k(f)\right) \mid S_t^k : a \le t \le b\right] = G^F_{\|f\|_{\Lambda_{k,T}^{sf}}, \left\|1_{(a,b)}f\right\|_{\Lambda_{k,T}^{sf}}}\left(\mathcal{I}_1^k\left(1_{(a,b)}f\right)\right). \tag{69}$$

Moreover, if $F \in C^1(R)$, $F' \in L^2\left(R, N(0, \|f\|_{\Lambda_{k,T}^{sf}}^2)\right)$, then the following relation holds

$$F\left(\mathcal{I}_1^k(f)\right) = E\left[\mathcal{I}_1^k(f)\right] + \delta_{k,T}^{ch}\left(G^{F'}_{\|f\|_{\Lambda_{k,T}^{sf}}, \left\|1_{(a,b)}f\right\|_{\Lambda_{k,T}^{sf}}}\left(\mathcal{I}_1^k\left(1_{(0,t)}f\right)\right) f\right). \tag{70}$$

Remarks. (a) Note that the *chaos sub-fractional Skorohod integral has zero expectation* (in contrast with the pathwise integral).
(b)Also, the Skorohod integral is the adjoint of the Malliavin derivative (the divergence operation).
(c) Sometimes the Skorohod integral is called the *Wick-Itô-Skorohod.* The reason is that in the case of Brownian motion the divergence coincides with the extension od the Itô integral introduced by Skorohod.
Also, one can show that under some regularity assumptions, we have

$$\delta_{k,T}^{ch}(u) = (L^2) \lim \sum_{t_k \in \pi} u_{t_{k-1}} \diamond \left(S_{t_k}^k - -S_{t_{k-1}}^k\right),$$

where $F \diamond G$ is so-called Wick product, so that the Shorohod integral can be considered as a limit of Riemann-Stieltjes sums if one replace the ordinary product by the Wick product.
A different approach to introduce the Skorohod integral uses the white noise analysis instead of Malliavin calculus (Bender).
(d) In fact, under regularity assumptions, the Riemann-Stieltjes integral is a Skorohod integral plus a "*Malliavin trace*".

Comment. In the case of fBm (which is older then of sfBm) the concepts and results are parallel. The approach with multiple integrals utilizes deterministic multiple Riemann-Liouville fractional integrals and derivatives instead of Erdélyi-Kober type multiple fractional integrals.

5 Sub-fractional Black-Scholes model

Next we consider the *sub-fractional Black-Scholes model.*
Comment. The case of *fractional Black-Scholes model* uses similar definitions and arguments.
In this model the bank account has the dynamics

$$dB_t = rB_t dt,\ 0 \leq t \leq T,\ B_0 = 1, \tag{71}$$

so that $B_t = \exp(rt)$ and the price of the risky asset has sub-fractional log normal dynamics

$$dS_t = \mu S_t dt + \sigma S_t dS_t^k,\ 0 \leq t \leq T,\ S_0 = s_0 > 0, \tag{72}$$

where μ is the *mean rate of return* and $\sigma > 0$ is the *volatility.*
If we interpret the stochastic integral in (72) as Riemann-Stieltjes, then by the change of variable formula (23) the solution of (72) is

$$S_t = s_0 \exp\left(\mu t + \sigma S_t^k\right). \tag{73}$$

As soon as we have the dynamics of the risky asset given by Riemann-Stieltjes integral, always this leads to the existence of arbitrage opportunities.

As example we recall the Shiryaev construction or an arbitrage for $\mu = r,\ \sigma = 1 : \pi = (u, v)$,

$$u_i = 1 - \exp(2S_t^k),\ v_t = 2\left[\exp(S_t^k) - 1\right].$$

From the change of variables formula (23) it follows that π is self-financing and moreover π is arbitrage, since

$$V_T^\pi = \left[\exp(S_T^k) - 1\right]^2 \exp(rT) > 0.$$

An alternative which works in the case of pathwise cost is to restrict the class of admissible portfolios, but to remain stil big enough to cover hedges for relevant options and also to consider mixed cost models.
In this respect we consider the *mixed model* with the stock price given by

$$S_t^{k,a} = s_0 \exp\left\{\sigma S_t^k + aW_t + \mu t - (1 - 2^{2k-1})\sigma^2 t^{2k+1} - \frac{1}{2}a^2 t\right\}, \qquad (74)$$

where $\sigma, s_0 > 0, a \in R,\ W$ is a Bm and the mixed process $\left(\sigma S_t^k + aW_t\right)_t$ is assumed to be Gaussian (this heapens, for example, if S^k and W are independent).

Remark. The mixed process $\left(\sigma S_t^k + aW_t\right)_t$ is a semimartingale equivalent with $(aW_t)_t$ if $k \in \left(\frac{1}{4}, \frac{1}{2}\right)$ and is not a semimartingale if $0 < k \leq \frac{1}{4}$.

The following class of restringed class of self-financing portfolios is considered: a self-financing portfolio $\pi = (u, v)$ is *nds-admissible* (no-doubling strategy) if there exists $a \geq 0$ such that $V_t^\pi \geq a$ for all $0 \leq t \leq T$ P-a.s.
A nds-admissible portfolio $\pi = (u, v)$ is regular if there exists a differentiable function $\varphi : [0, T] \times R_+^4 \longrightarrow R$ such that

$$v_t = \varphi\left(t, S_t^{k,a}, \max_{0 \leq s \leq t} S_s^{k,a}, \min_{0 \leq s \leq t} S_s^{k,a}, \int_0^t S_s^{k,a} ds\right).$$

As a consequence of a more general result due to Bender-Sottined-Valkeila (Finance Stoch., 2008) the following result holds:

Theorem 5.1. *The mixed model is arbitrage free in the class of regular portfolios.*
Moreover, European, Assian, lookback options can be hedged with regular portfolios, with the same functionals and hedging prices as in tht classical Black-Scholes model.

A different alternative to the pathwise approach of the stock price is to consider the chaos form of the stochastic integral in (72).
We shall use the notation $\int_0^t f(s)\delta_{k,T}^{ch} B_s^H$ for the sub-fractional Skorohod integral $\delta_{k,T}^{ch}\left(1_{(0,t)}f\right)$.
Therefore the price of the risky asset has the dynamics

$$dS_t = \mu S_t dt + \sigma S_t \delta_{k,T}^{ch} S_t^k,\ 0 \leq t \leq T,\ S_0 = s_0 > 0. \tag{75}$$

In the present situation the portofolio π is *self-financing* if $v_. S \in L^1([0,T])$, $1_{(0,t)} vS \in \left|S^{ch}\right|_{k,T}^{sf}$ for a.a. t, and

$$V_t^\pi = V_0^\pi + \int_0^t (ru_s \exp(rs) + \mu v_s S_s)\, ds + \sigma \int_0^t v_s S_s \delta_{k,T}^{ch} S_s^k,\ 0 \leq t \leq T. \tag{76}$$

A self-financing portofolio π is *admissible* if V_t^π is bounded below for all $0 \leq t \leq T$.
In order to find the explicit form of the S_t in (75) we consider the following sub-fractional affine equation

$$X_t = \eta + \int_0^t [a_0(s) + a(s)X_s]\, ds + \int_0^t [b_0(s) + b(s)X_s]\, \delta_{k,T}^{ch} B_s^H,\ t \in [0,T], \tag{77}$$

$\eta \in L^2(\Omega, \mathcal{F}, P)$ and $a_0, b_0, a, b : [0,T] \to R$ are measurable and bounded functions.

Definition. A process $(X_t)_{t\in[0,T]}$ is a *strong solution* of (77) if
(i) $X_. \in L^1([0,T])$ and $1_{[0,t]}(.)b(.)X_.$ is sub-fractional Skorohod integrable for a.a. $t \in [0,T]$.
(ii) For a.a. $t \in [0,T]$, the equation (77) is satisfied $P-$a.s..

Utilizing the chaos decomposition we have

Proposition 5.2. *Assume that $\eta = \eta_0 \in R$ and define*

$$\Phi(t,s) = \exp\left\{ \int_s^t a(u)du + \int_s^t b(u)dB_u^H - \frac{1}{2}\left\| b1_{[s,t]} \right\|^2_{\Lambda^{sf}_{k,T}} \right\}.$$

Then (77) has a unique strong solution X

$$X_t = \Phi(t,t_0)\eta_0 + \int_{t_0}^t \Phi(t,s)a_0(s)ds + \int_{t_0}^t \Phi(t,s)b_0(s)dB_s^H \qquad (78)$$

Corollary. *The stock price S_t in* (75) *is given by*

$$S_t = s_0 \exp\left\{ \mu t - \left(1 - 2^{2k-1}\right)\sigma^2 t^{2k+1} + \sigma S_t^k \right\}. \qquad (79)$$

Remark. The standard Black-Scholes model is markovian and the log-returns are stationary independent Gaussian random variables.
The fractional Black-Scholes model is nonmarkovian and the log-returns are stationary non-independent Gaussian random variables.
The sub-fractional Black-Scholes model differs from fractional Black-Scholes model by the *non stationarity* of the log-returns

$$R_{t,t+s} = \mu s - \left(1 - 2^{2k-1}\right)\sigma^2\left[(t+s)^{2k+1} - t^{2k+1}\right] + \sigma\left(S^k_{t+s} - S^k_t\right).$$

Definition. A probability measure Q on $\mathcal{F}_T^S$ which is equivalent with P ($Q \sim P$) is called a *quasi-martingale measure* (or *average risk neutral measure*) if:

(i) There exists a Gaussian process $(Z_t)_{0\le t\le T}$ with respect to Q such that

$$S_t B_t^{-1} = \exp(Z_t), 0 \le t \le T. \qquad (80)$$

(ii) For every $0 \le t \le T$,

$$E_Q(S_t B_t^{-1}) = s_0. \qquad (81)$$

Remark. It is clear that if Q is a quasi-martingale measure then Q is uniquely determined on $\mathcal{F}_T^{S^k} = \mathcal{F}_T^S$.

Define the probability measure Q by

$$\frac{dQ}{dP}\mid_{\mathcal{F}_t^{S^k}} = \exp\left\{ -\frac{\mu - r}{\sigma}M_t^k - \left(\frac{\mu - r}{\sigma}\right)^2 \frac{d_k^2}{2(1-2k)} t^{1-2k} \right\}. \qquad (82)$$

Remarks. (a) By Girsanov's theorem, the process

$$\left(Z_t^k := S_t^k + \frac{\mu - r}{\sigma} t\right)_t$$

is a sfBm under Q.

(b) The relation stock price becomes in terms of Z^k

$$S_t = \exp\left\{Z_t^k - (1 - 2^{2k-1}) t^{2k+1}\right\}, \tag{83}$$

and it is clear that S_t has under Q the dynamics

$$dS_t = S_t \delta_{k,T}^{ch} Z_t^k, 0 \leq t \leq T, \ S_0 = s_0. \tag{84}$$

The main result is the following

Theorem 5.3. *(i) The sub-fractional Black-Scholes market is arbitrage free and for every bounded contigent claim* $F \in \left|D_k^{1,2}(Q)\right|$ *there exist* $v_0 \in R$ *and an admissible portofolio* π *such that*

$$V_0^\pi = v_0, \; V_T^\pi = F \; P - a.s.$$

(ii) The following relation holds:

$$E_Q\left[S_T B_T^{-1} \mid \mathcal{F}_t^{S^k}\right] = S_t B_t^{-1} \exp(K(T,t)), \; \forall 0 \le t \le T, \qquad (85)$$

where

$$K(T,t) = \exp\left\{d_k(r-\mu) \int_0^t I^k_{T-,2,\frac{k+1}{2}}\left(1_{(0,t)}\Psi_{t,T}\right)(s)\, s^{-k} ds\right.$$

$$\left.-\sigma^2 c_k^2 \int_0^t n^2(T,s) ds + \sigma \int_0^t \Psi_{t,T}(s) dS_s^k + \sigma^2(1-2^{2k-1}) t^{2k+1}\right. . \qquad (86)$$

In particular Q is the unique quasi-martingale measure and Q is not a martingale measure.

Remark. The *price* $C_T(F)$ of the contigent claim F is given by

$$C_T(F) = E_Q\left(B_T^{-1} F\right). \qquad (87)$$

The corresponding replicating portfolio $\pi = (u,v)$ also can be described. The price of the contigent clain $f(S_T)$ is given by the formula

$$C_T(f(S_T)) = \frac{\exp(-rT)}{\sqrt{2\pi}}$$

$$\times \int_R f s_0 \exp\left\{\left(\sigma T^{k+\frac{1}{2}} y + rT - \sigma^2\left(1-2^{2k-1}\right)\right) T^{2k+1}\right\} \exp(-\frac{y^2}{2}) dy \qquad (88)$$

In particular the price of of an European call is

$$C_T((S_T-K)^+) = s_0 \Phi(y_1) - K \exp(-rT)\Phi(y_2), \qquad (89)$$

$$y_1 = \frac{\log \frac{s_0}{K} + rT + \sigma^2\left(1-2^{2k-1}\right) T^{2k+1}}{\sigma\sqrt{2-2^{2k}} T^{k+\frac{1}{2}}},$$

$$y_1 = \frac{\log \frac{s_0}{K} + rT - \sigma^2 \left(1 - 2^{2k-1}\right) T^{2k+1}}{\sigma\sqrt{2 - 2^{2k}} T^{k+\frac{1}{2}}}.$$

Comments. (a) In the mixed sub-fractional model the class of regular portfolios is a abitrage-free class that is sufficiently large to cover hedges for most known relevant options.

A recent result by Bender, Sottinen and Valkeila (Finance Stoch., 2008) extends no-arbitrage property and robust hedges to a class of non-semimartingale models larger than the mixed processes and a larger class of portfolios.

It should be noted that the quadratic variation is the main property which is necessary for pricing in non-semimartingale models.

(b) The use of chaos form of the Skorohod integral in the sub-fractional Black-Scholes model does not have a nice economic interpretation (Bjork-Hult, 2005 for fBm case) and also this is problematic from the mathematical point of view (Nualart-Taqqu, 2008). It happens that different Gaussian processes with the same variation as S^k give the same price for European call options.

Therefore in above mentioned both cases *it is not the distribution of the process* which determines uniquely the prices, but *the variation of the process.*

(c) An alternative to rescue the sub-fractional Black-Scholes model is to use so called *market observers* according to an idea by Øksendal (Bender, 2003)

6 A decomposition of sub-fractional Brownian motion

Recall that a continuous process $(X_t)_{t\in[0,T]}$ admits $\alpha-$*variation* (resp. $\alpha-$*strong variation*) if the following limit in probability exists for every $t \in [0,T]$,

$$V_t^{n,\alpha}(X) = \sum_{i=0}^{n-1} \left| X_{\frac{(i+1)t}{n}} - X_{\frac{it}{n}} \right|^{\alpha},$$

resp.

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t |X_{s+\varepsilon} - X_s|^{\alpha} \, ds.$$

Remark 6.1. For the fBm it is known that $\frac{2}{2k+1}-$variation (resp. $\frac{2}{2k+1}$-strong variation) is $\rho_{\frac{2}{2k+1}} t$, where $\rho_p = E\left(|N(0,1)|^p\right)$(see *Rogers-Math. Finance-1997* for the case of variation : the case of strong variation follows along the same arguments as for variation).
For sfBm the same result is obtained by using linear regression (see *CT-Stochastics-2007*).
We consider the function

$$K(s,t) = \frac{1}{2}\left[s^{2k+1} + t^{2k+1} - (s+t)^{2k+1}\right], \; s,t \in [0,T], \tag{90}$$

and if $(W_t)_{t\in[0,T]}$ is a Brownian motion, we define the process $\left(X_t^k\right)_{t\in[0,T]}$ as the Wiener integral

$$X_t^k = \int_0^{\infty} \left(1 - e^{-\theta t}\right) \theta^{-k-1} dW_{\theta}. \tag{91}$$

Remark 6.2 (*Nualart-Lei: SPL 79, 2009*). The centered Gaussian process $\left(X_t^k\right)_{t\in[0,T]}$ has the covariance

$$C_{X^k}(s,t) = -\frac{\Gamma(1-2k)}{k(2k+1)} K(s,t), \tag{92}$$

and has the representation

$$X_t^k = \int_0^{\infty} Y_t^k dt, \tag{93}$$

$$Y_t^k = \int_0^\infty e^{-\theta t}\theta^{-k} dW_\theta. \tag{94}$$

In particular X^k has a version with infinitely differentiable paths on $(0,\infty)$ and absolutely continuous paths on R_+.

Remark 6.3. Note that K is a covariance function if $k \in \left(-\frac{1}{2}, 0\right)$ and $-K$ is also a covariance function if $k \in \left(0, \frac{1}{2}\right)$.

Denote

$$|\Lambda|_{X^k} = \left\{ f : [0,T] \to R : \int_0^T \int_0^T |f(s)f(t)| \, (s+t)^{2k-1} ds dt < \infty \right\}.$$

Lemma 6.4. *We have the inclusion $|\Lambda|_{X^k} \subset \Lambda_{X^k}$ and the relation* (denoting by $\|f\|_{\Lambda_{X^k}}$ the norm of f as an element of the domain of the Wiener integral Λ_{X^k})

$$\|f\|^2_{\Lambda_{X^k}} = \Gamma(1-2k) \int_0^T \int_0^T f(s)f(t)(s+t)^{2k-1} ds dt, f \in |\Lambda|_{X^k}. \tag{95}$$

Moreover, if $f \in L^1\left([0,T], t^{k-\frac{1}{2}} dt\right)$ then $f \in |\Lambda|_{X^k}$ and the following equality holds

$$\int_0^T f(t) dX_t^k = \int_0^T f(t) Y_t^k dt. \tag{96}$$

Theorem 6.5 (J.R. de Chavez, CT-2009) (a) *Let* $k \in \left(-\frac{1}{2}, 0\right)$ *and let* $\left(B_t^k\right)_{t\in[0,T]}$ *be a fBm independent of the Bm* $(W_t)_{t\in[0,T]}$.
Then the process

$$S_t^k = \sqrt{-\frac{k(2k+1)}{\Gamma(1-2k)}}X_t^k + B_t^k, t \in [0,T], \tag{97}$$

is a sfBm. In particular

$$\Lambda_{X^k} \cap \Lambda_{B^k} = \Lambda_{S^k}. \tag{98}$$

Moreover, if

$$f \in I_{T-}^{-k}\left(L^2([0,T])\right) \cap L^1\left([0,T], t^{k-\frac{1}{2}}dt\right),$$

then $f \in \Lambda_{S^k}$ *and*

$$\int_0^T f(t)dS_t^k = \sqrt{-\frac{k(2k+1)}{\Gamma(1-2k)}}\int_0^T f(t)Y_t^k dt + \int_0^T f(t)dB_t^k, \tag{99}$$

$$\begin{aligned}\left\|\int_0^T f(t)dS_t^k\right\|_{L^2(\Omega,\mathcal{F},P)}^2 &= \Gamma(1-2k)\int_0^T\int_0^T f(s)f(t)(s+t)^{2k-1}dsdt \\ &+\frac{\pi k(2k+1)}{\Gamma(1-2k)\sin\pi k}\|\varphi\|_{L^2([0,T])}^2,\end{aligned} \tag{100}$$

where $I_{T-}^{-k}(s^k\varphi)(u) = u^k f(u)$.
(b) *Let* $k \in \left(0, \frac{1}{2}\right)$ *and let* $\left(S_t^k\right)_{t\in[0,T]}$ *be a sfBm independent of the Bm* $(W_t)_{t\in[0,T]}$.
Then the process

$$B_t^k = \sqrt{\frac{k(2k+1)}{\Gamma(1-2k)}}X_t^k + S_t^k, t \in [0,T], \tag{101}$$

is a fBm. In particular

$$\Lambda_{X^k} \cap \Lambda_{S^k} = \Lambda_{B^k}. \tag{102}$$

Moreover, if

$$\int_0^T\int_0^T |f(s)f(t)||s-t|^{2k-1}dsdt < \infty, \tag{103}$$

then $f \in \Lambda_{B^k}$ *and*

$$\int_0^T f(t) dS_t^k = \int_0^T f(t) dB_t^k - \sqrt{\frac{k(2k+1)}{\Gamma(1-2k)}} \int_0^T f(t) Y_t^k dt, \tag{104}$$

$$\begin{aligned}\left\| \int_0^T f(t) dS_t^k \right\|_{L^2(\Omega,\mathcal{F},P)}^2 &= k(2k+1) \int_0^T \int_0^T f(s) f(t) \left| s-t \right|^{2k-1} dsdt \\ &-\Gamma(1-2k) \int_0^T \int_0^T f(s) f(t) (s+t)^{2k-1} dsdt. \end{aligned} \tag{105}$$

Remark 6.6. In *Monrad-Rootzén: PTRF-1995* the following Chung's law of iterated logariihm is obtained for fBm: For every $t_0 > 0$ there exists a positive constant $c_k(t_0)$ such that

$$\lim_{t \to 0} \frac{\max_{0 \le r \le t} \left| B_{r+t_0}^k - B_{t_0}^k \right|}{t^{\frac{2k+1}{2}} \left(\log \left| \log t \right| \right)^{\frac{2k+1}{2}}} = c_k(t_0), \ a.s.$$

Proposition 6.7. (a) *The* $\frac{2}{2k+1}$*−variation* (*resp.* $\frac{2}{2k+1}$*−strong variation*) *of sfBm is* $\rho_{\frac{2}{2k+1}} t$.
(b) (*Chung's law of iterated logarithm for sfBm*). *We have*

$$\lim_{t \to 0} \frac{\max_{0 \le r \le t} \left| S_r^k \right|}{t^{\frac{2k+1}{2}} \left(\log \left| \log t \right| \right)^{\frac{2k+1}{2}}} = c_k(t_0), \ a.s.$$

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