Inverse Problems in Solid Mechanics

a short review

Andrei Constantinescu

Laboratoire de Mécanique des Solides
CNRS UMR 7649
Département de Mécanique, Ecole Polytechnique

Palaiseau, France

Bucharest, september 2010

Inverse Problem

- **Inverse problems in elasticity**
  - General framework
  - Identification of cracks using the reciprocity gap
  - Identification of distributed elastic moduli
    error in constitutive law
- **Inverse problems and sensibility computations**
  - Sensibility and contact boundary conditions
  - Identification of parameters of constitutive laws:
    - Identation problem
    - Poroelasticity
- **Conclusion & Perspectives**

Inverse Problem

- **Inverse problems in elasticity**
  - General framework
  - Identification of cracks using the reciprocity gap
  - Identification of distributed elastic moduli
    error in constitutive law
- **Inverse problems and sensibility computations**
  - Sensibility and contact boundary conditions
  - Identification of parameters of constitutive laws:
    - Identation problem
    - Poroelasticity
- **Conclusion & Perspectives**

Linear Elasticity (small strains)

Direct Problem $\mathcal{P}$ $\qquad \boldsymbol{\sigma}(\boldsymbol{x},t) = \boldsymbol{C} : \boldsymbol{\varepsilon}[\boldsymbol{u}]$

Balance of forces

$$\mathrm{div}\,\boldsymbol{\sigma}[\boldsymbol{u}] + \boldsymbol{f} - \rho\,\ddot{\boldsymbol{u}} = 0$$

initial conditions $\qquad (\boldsymbol{x} \in \Omega)$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \dot{\boldsymbol{u}}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x})$$

boundary conditions $\qquad (\boldsymbol{x},t) \in S_u \cup S_p \times [0,T]$

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) &= \boldsymbol{u}^D(\boldsymbol{x},t) \\ \boldsymbol{\sigma}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) &= \boldsymbol{p}^D(\boldsymbol{x},t) \end{aligned}$$

Inverse Problems *[Bonnet & AC, Inv.Problems 2005]*

- identification of constitutive parameters *[AC, Grediac & Pierron]*
- identification of boundary tractions *[Cimetière et al]*
- reconstruction of an inclusion *[Nemitz & Bonnet, 2005]*

Techniques

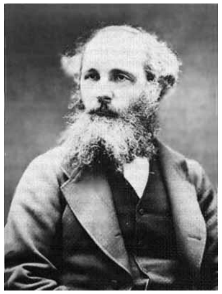
- variational formulation: principal of virtual power
- strain-stress duality
- minimisation of a cost functional:

$$p = \underset{q,u\in\mathcal{P}q}{\arg\min}\, \mathcal{J}(\boldsymbol{u},q)$$

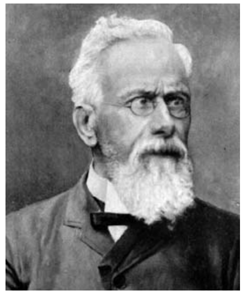
Reciprocity

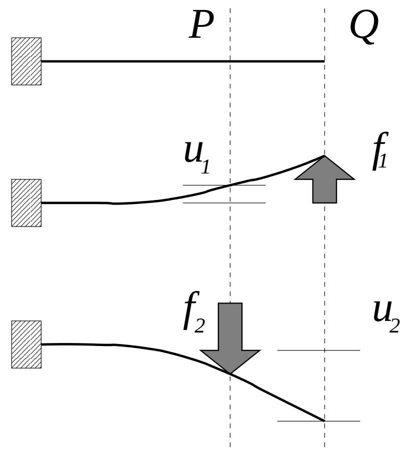
M.Faraday
1791 – 1867

J.C.Maxwell
1831 – 1879



E.Betti
1823 – 1892





Problème

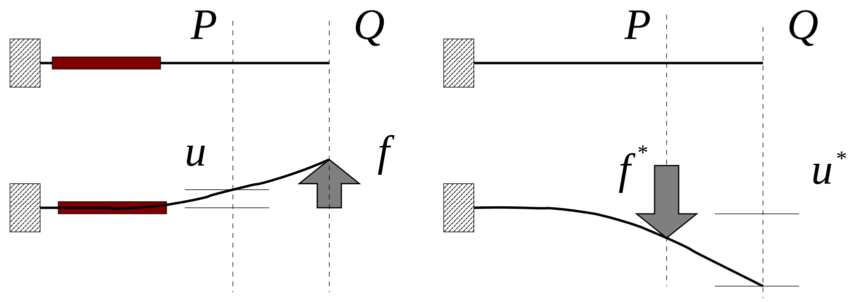
$$\boldsymbol{A} \cdot \boldsymbol{u} = \boldsymbol{f} \qquad \begin{bmatrix} A_{PP} & A_{PQ} \\ A_{QP} & A_{QQ} \end{bmatrix} \cdot \begin{bmatrix} u_P \\ u_Q \end{bmatrix} = \begin{bmatrix} f_P \\ f_Q \end{bmatrix}$$

Maxwell-Betti Reciprocity Theorem

$$\boldsymbol{f}_2 \cdot \boldsymbol{u}_1 = \boldsymbol{f}_1 \cdot \boldsymbol{u}_2$$

- existence of internal energy
- symetry of elatic moduli $[\boldsymbol{A}]$

Réciprocité



Problème direct

$$(\boldsymbol{A} + \delta \boldsymbol{A}) \cdot \boldsymbol{u} \quad = \quad \boldsymbol{f}$$

Problème adjoint

$$\boldsymbol{A} \cdot \boldsymbol{u}^* \quad = \quad \boldsymbol{f}^*$$

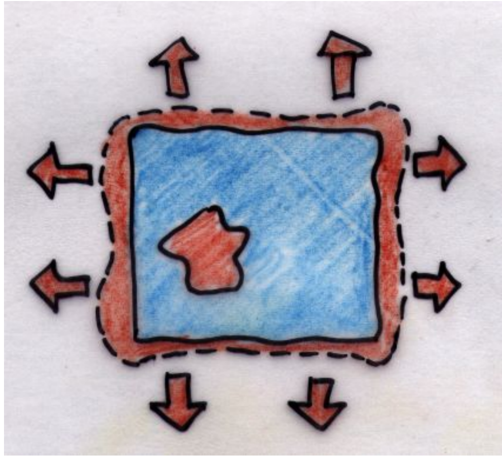
Principe de Maxwell

$$\boldsymbol{f}^* \cdot \boldsymbol{u} \neq \boldsymbol{f} \cdot \boldsymbol{u}^*$$

Ecart à la réciprocité

$$\begin{aligned} \mathcal{R} &= \boldsymbol{u}_1^\top \cdot \boldsymbol{f}_2^* - \boldsymbol{u}_2^{*T} \cdot \boldsymbol{f}_1 \\ &= \boldsymbol{u}_2^{*T} \cdot (\delta \boldsymbol{A}) \cdot \boldsymbol{u}_1. \end{aligned}$$

Reciprocity & Identifiability of $C^*(x)$



Equilibrium

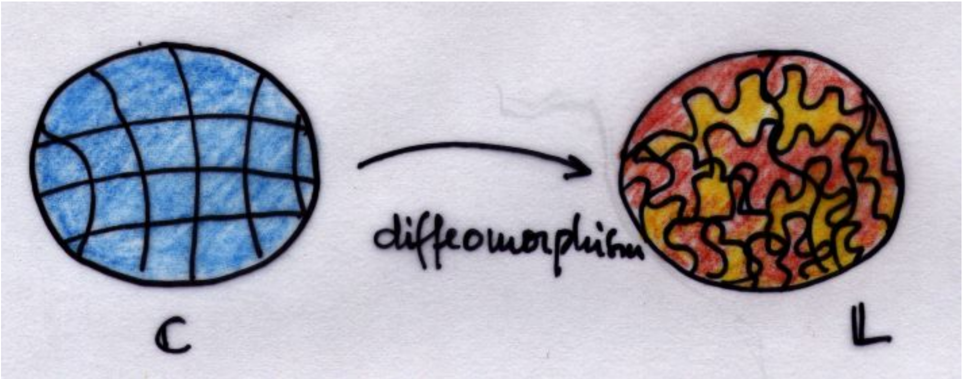
$$\text{div}\, \boldsymbol{C} : \varepsilon[\boldsymbol{u}] = 0$$

boundary conditions $\boldsymbol{x} \in S$

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}) &= \boldsymbol{u}^D(\boldsymbol{x}) \\ \boldsymbol{\sigma}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) &= \boldsymbol{p}^D(\boldsymbol{x}) \end{aligned}$$

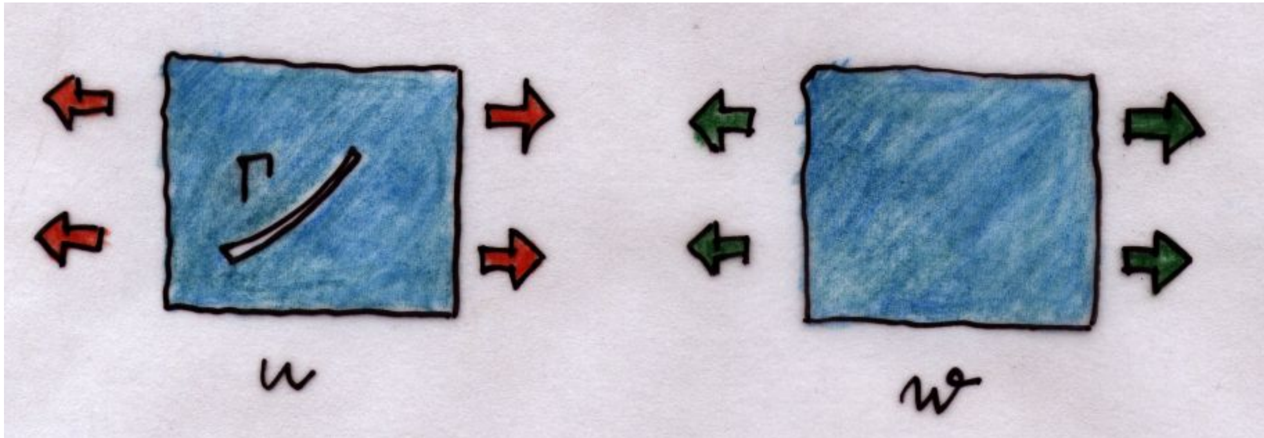
$$\int_S \boldsymbol{u} \cdot \boldsymbol{p}_C[\boldsymbol{w}]\, ds = \int_\Omega \nabla \boldsymbol{u} : \boldsymbol{C} : \nabla \boldsymbol{w}\, dv$$

Nonuniqueness for anisotropic $C^*(x)$



$$L_{ijk\ell}(\boldsymbol{\Psi}(\boldsymbol{x})) = |\text{det}\nabla\boldsymbol{\Psi}|^{-1}(\boldsymbol{x}) C_{imkn}(\boldsymbol{x}))\Psi_{j,m}(\boldsymbol{x}))\Psi_{\ell,n}(\boldsymbol{x}))$$

Reciprocity - Crack Identification



Direct Problem

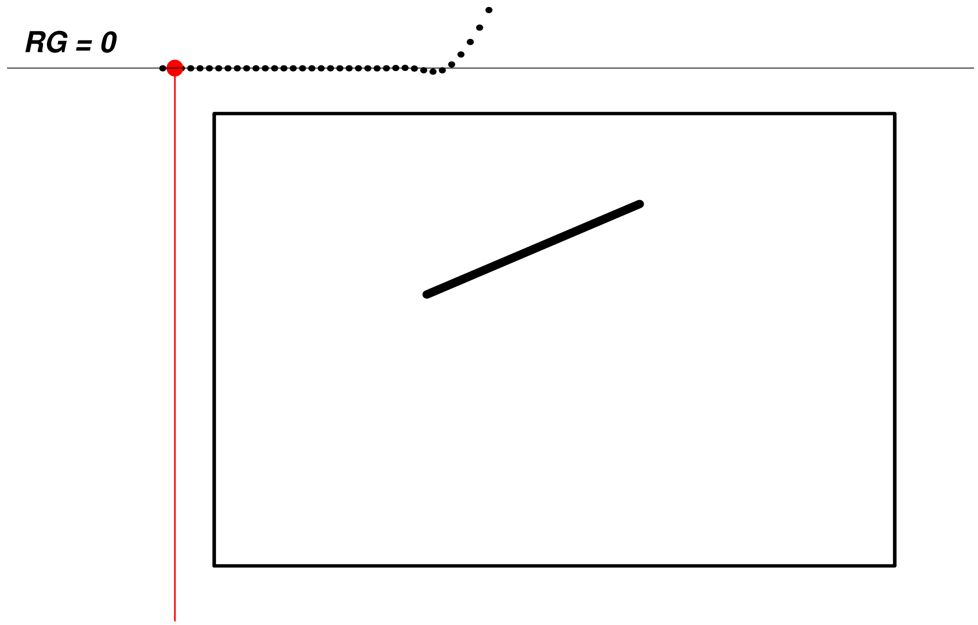
$$\mathrm{div}\, \boldsymbol{C}\, \nabla \boldsymbol{u} \;=\; \rho\, \ddot{\boldsymbol{u}} \qquad\qquad \mathrm{div}\, \boldsymbol{C}\, \nabla \boldsymbol{w} \;=\; \rho\, \ddot{\boldsymbol{w}}$$

Reciprocity Gap

$$\begin{aligned} \mathcal{RB}(\boldsymbol{u}, \boldsymbol{w}, \Gamma) &= \int_0^{\infty} \int_{\Gamma} [\![\boldsymbol{u}]\!] \cdot \boldsymbol{\sigma}[\boldsymbol{w}] \cdot \boldsymbol{n}\, ds\, dt \\ &= \int_0^{\infty} \int_{\partial\Omega} \{\boldsymbol{u} \cdot \boldsymbol{\sigma}[\boldsymbol{w}] \cdot \boldsymbol{n} - \boldsymbol{w} \cdot \boldsymbol{\sigma}[\boldsymbol{u}] \cdot \boldsymbol{n}\}\, ds\, dt + \int_{\Omega \setminus \Gamma} [\boldsymbol{u} \cdot \partial_t \boldsymbol{w} - \partial_t \boldsymbol{u} \cdot \boldsymbol{w}]_0^{\infty}\, dv \end{aligned}$$

- electrostatics & thermal diffusion: S. Andrieux & A. Ben Abda [1996], H.D.Bui & A.Ben Abda [1998]
- acoustics, elastodynamics: H.D.Bui, AC, H.Maigre [1999,2004,2005]
- Helmholtz equation - enclosure method: Ikehata [2000,. . .]

Ecart à la réciprocité: identification de la faille

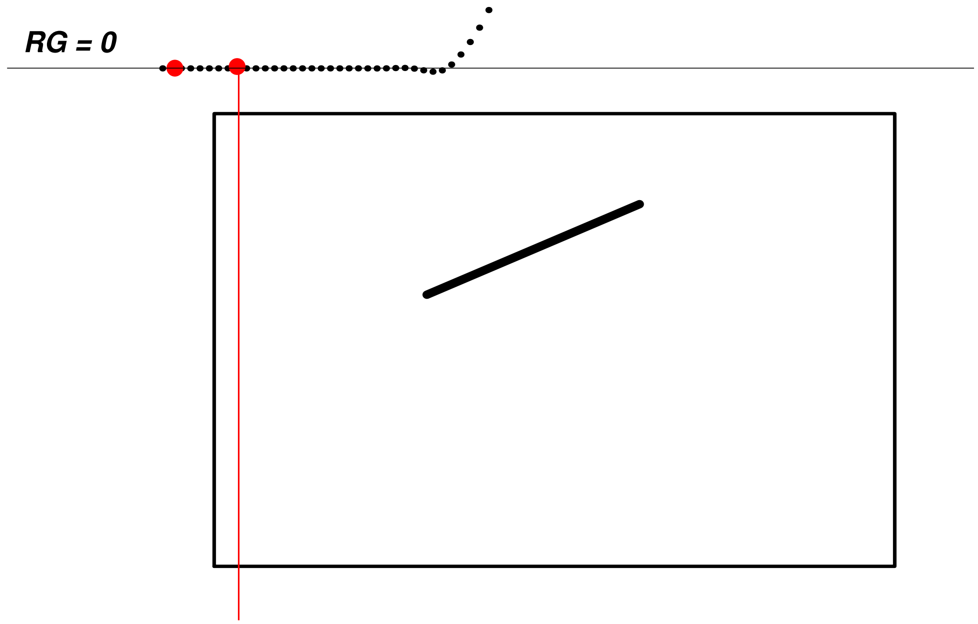


Réciprocité instantané

$$\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{q}H(t-\boldsymbol{x}\cdot\boldsymbol{p}) \qquad \boldsymbol{\sigma}[\boldsymbol{w}]\cdot\boldsymbol{n} = \boldsymbol{\tau}[\boldsymbol{w}]\delta(t-\boldsymbol{x}\cdot\boldsymbol{p})$$

$$\mathcal{R}(t) = \int_\Gamma [\![\boldsymbol{u}^t]\!]\cdot\boldsymbol{\tau}[\boldsymbol{w}]\,ds = \int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{\tau}[\boldsymbol{w}]\cdot\boldsymbol{n}\,ds$$

Ecart à la réciprocité: identification de la faille

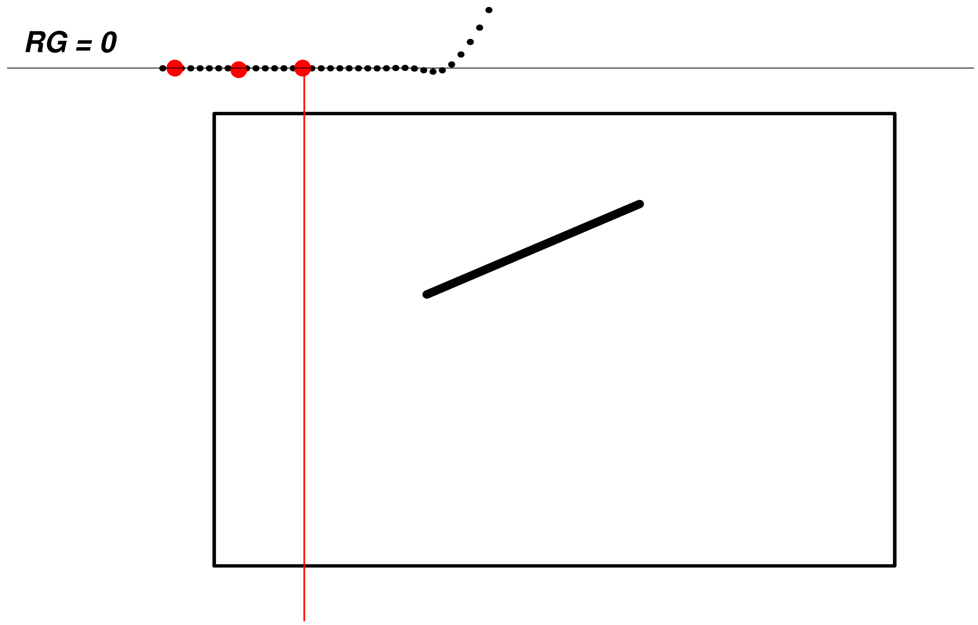


Réciprocité instantané

$$\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{q}H(t - \boldsymbol{x}\cdot\boldsymbol{p}) \qquad \boldsymbol{\sigma}[\boldsymbol{w}]\cdot\boldsymbol{n} = \boldsymbol{\tau}[\boldsymbol{w}]\delta(t - \boldsymbol{x}\cdot\boldsymbol{p})$$

$$\mathcal{R}(t) = \int_{\Gamma} [\![\boldsymbol{u}^t]\!]\cdot\boldsymbol{\tau}[\boldsymbol{w}]\,ds = \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{\tau}[\boldsymbol{w}]\cdot\boldsymbol{n}\,ds$$

Ecart à la réciprocité: identification de la faille

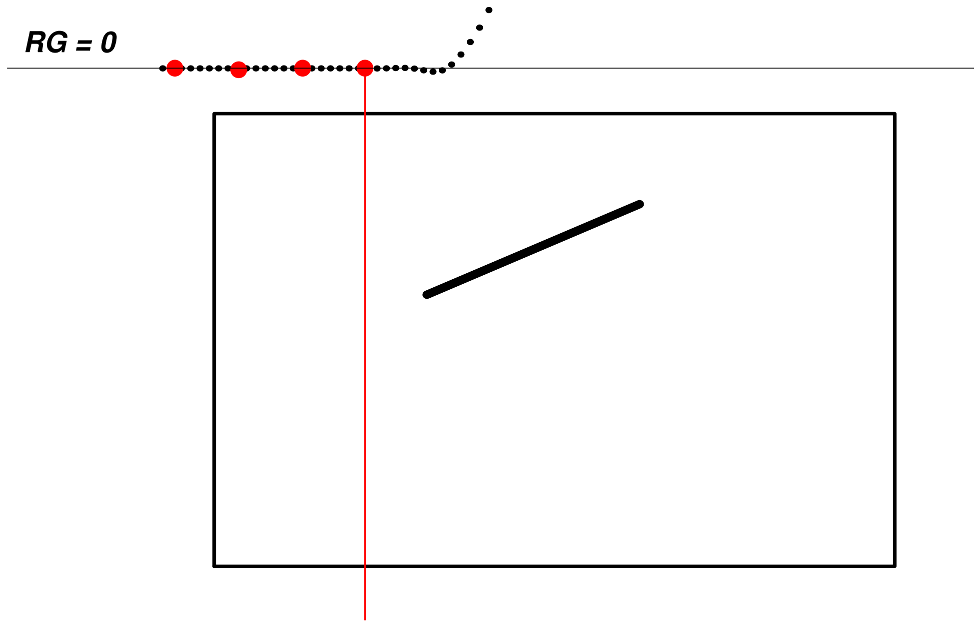


Réciprocité instantané

$$\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{q}H(t - \boldsymbol{x}\cdot\boldsymbol{p}) \qquad \boldsymbol{\sigma}[\boldsymbol{w}]\cdot\boldsymbol{n} = \boldsymbol{\tau}[\boldsymbol{w}]\delta(t - \boldsymbol{x}\cdot\boldsymbol{p})$$

$$\mathcal{R}(t) = \int_\Gamma [\![\boldsymbol{u}^t]\!]\cdot\boldsymbol{\tau}[\boldsymbol{w}]\,ds = \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{\tau}[\boldsymbol{w}]\cdot\boldsymbol{n}\,ds$$

Ecart à la réciprocité: identification de la faille

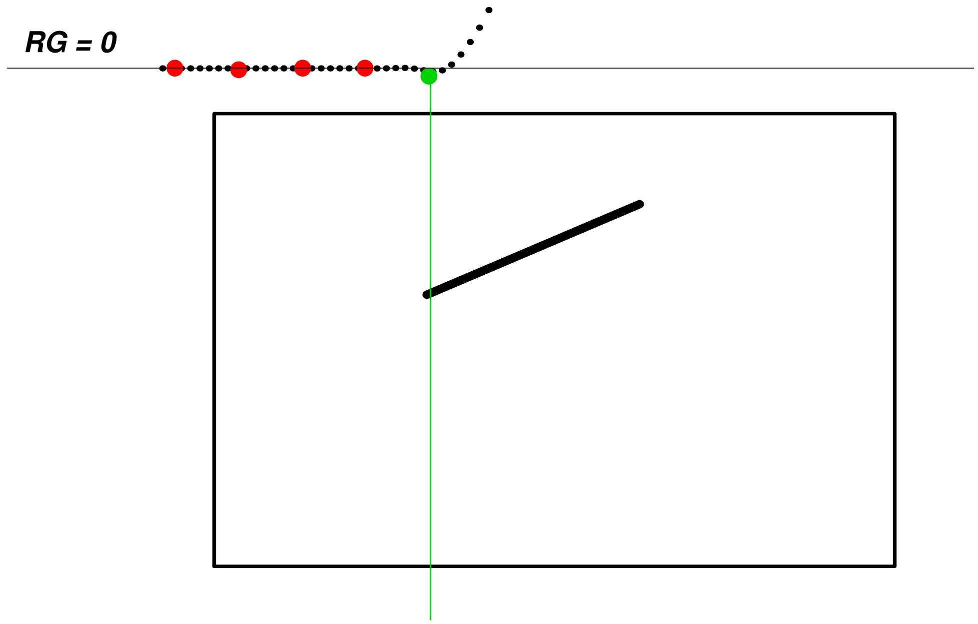


Réciprocité instantané

$$\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{q}H(t - \boldsymbol{x}\cdot\boldsymbol{p}) \qquad \boldsymbol{\sigma}[\boldsymbol{w}]\cdot\boldsymbol{n} = \boldsymbol{\tau}[\boldsymbol{w}]\delta(t - \boldsymbol{x}\cdot\boldsymbol{p})$$

$$\mathcal{R}(t) = \int_{\Gamma} [\![\boldsymbol{u}^t]\!]\cdot\boldsymbol{\tau}[\boldsymbol{w}]\,ds = \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{\tau}[\boldsymbol{w}]\cdot\boldsymbol{n}\,ds$$

Ecart à la réciprocité: identification de la faille

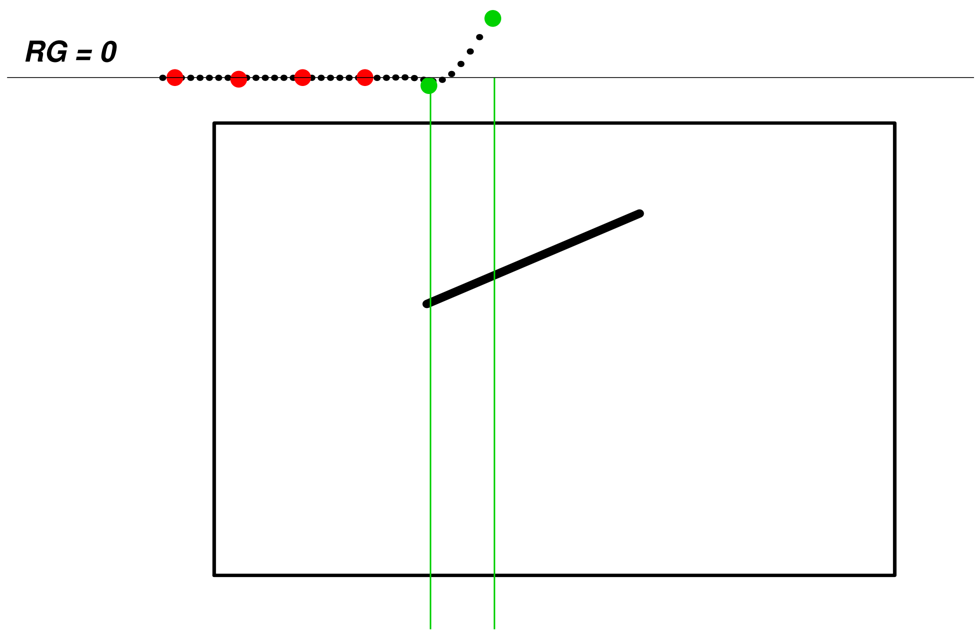


Réciprocité instantané

$$\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{q}H(t - \boldsymbol{x}\cdot\boldsymbol{p}) \qquad \boldsymbol{\sigma}[\boldsymbol{w}]\cdot\boldsymbol{n} = \boldsymbol{\tau}[\boldsymbol{w}]\delta(t - \boldsymbol{x}\cdot\boldsymbol{p})$$

$$\mathcal{R}(t) = \int_\Gamma [\![\boldsymbol{u}^t]\!]\cdot\boldsymbol{\tau}[\boldsymbol{w}]\,ds = \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{\tau}[\boldsymbol{w}]\cdot\boldsymbol{n}\,ds$$

Ecart à la réciprocité: identification de la faille

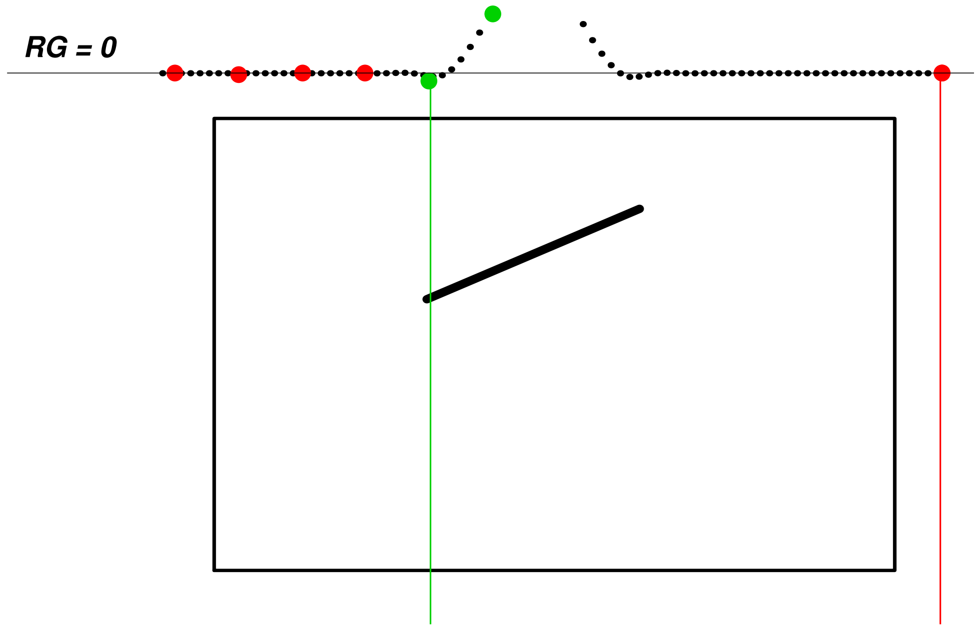


Réciprocité instantané

$$\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{q} H(t - \boldsymbol{x}\cdot\boldsymbol{p}) \qquad \boldsymbol{\sigma}[\boldsymbol{w}]\cdot\boldsymbol{n} = \boldsymbol{\tau}[\boldsymbol{w}]\delta(t - \boldsymbol{x}\cdot\boldsymbol{p})$$

$$\mathcal{R}(t) = \int_{\Gamma} [\![\boldsymbol{u}^t]\!]\cdot\boldsymbol{\tau}[\boldsymbol{w}]\,ds = \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{\tau}[\boldsymbol{w}]\cdot\boldsymbol{n}\,ds$$

Ecart à la réciprocité: identification de la faille

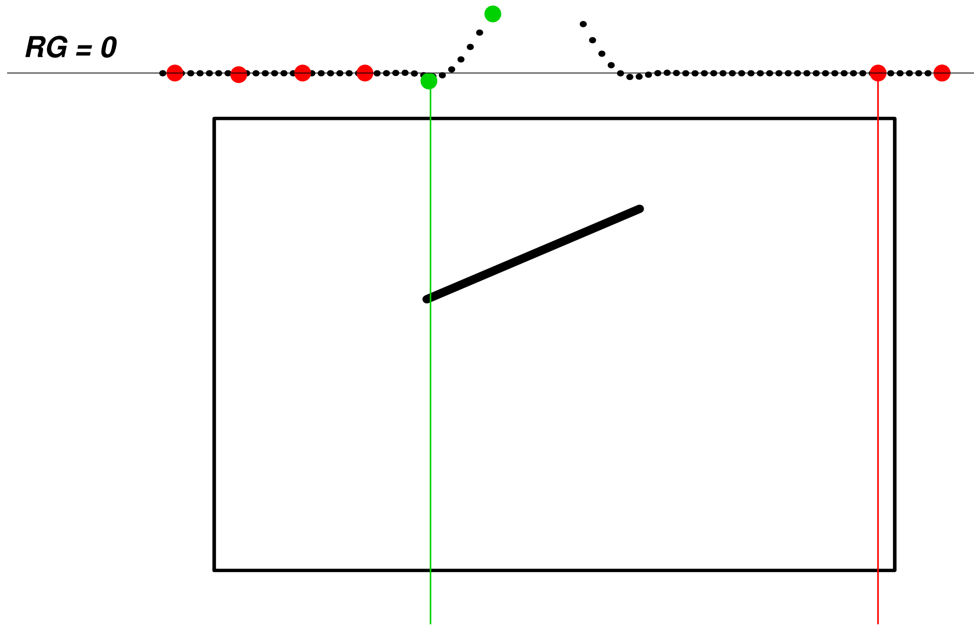


Réciprocité instantané

$$w(x,t) = qH(t - x \cdot p) \qquad \sigma[w] \cdot n = \tau[w]\delta(t - x \cdot p)$$

$$\mathcal{R}(t) = \int_\Gamma [\![u^t]\!] \cdot \tau[w]\, ds = \int_{\partial\Omega} u \cdot \tau[w] \cdot n\, ds$$

Ecart à la réciprocité: identification de la faille

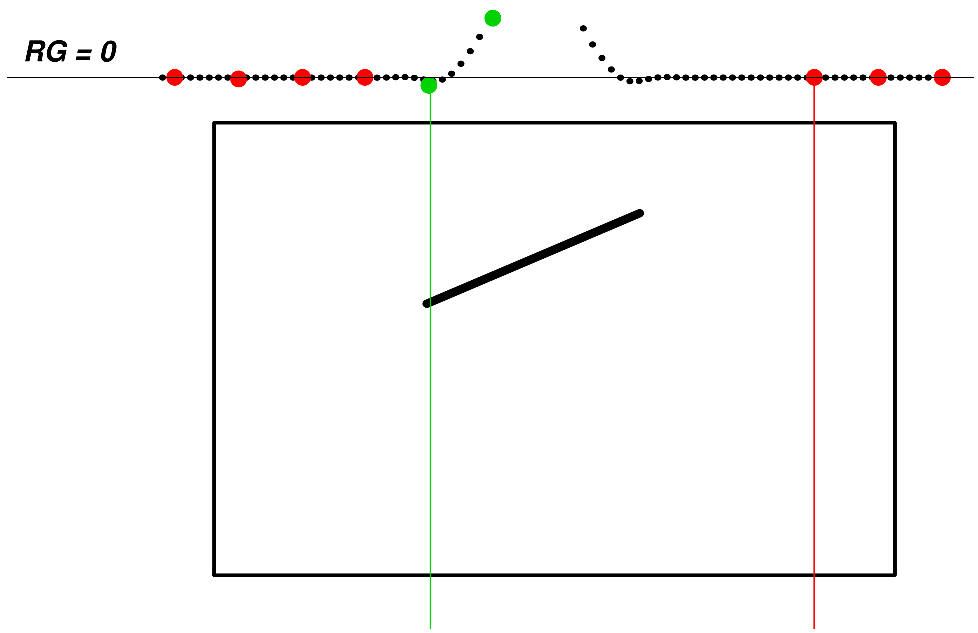


Réciprocité instantané

$$\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{q}H(t - \boldsymbol{x}\cdot\boldsymbol{p}) \qquad \boldsymbol{\sigma}[\boldsymbol{w}]\cdot\boldsymbol{n} = \boldsymbol{\tau}[\boldsymbol{w}]\delta(t - \boldsymbol{x}\cdot\boldsymbol{p})$$

$$\mathcal{R}(t) = \int_{\Gamma} [\![\boldsymbol{u}^t]\!]\cdot\boldsymbol{\tau}[\boldsymbol{w}]\,ds = \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{\tau}[\boldsymbol{w}]\cdot\boldsymbol{n}\,ds$$

Ecart à la réciprocité: identification de la faille

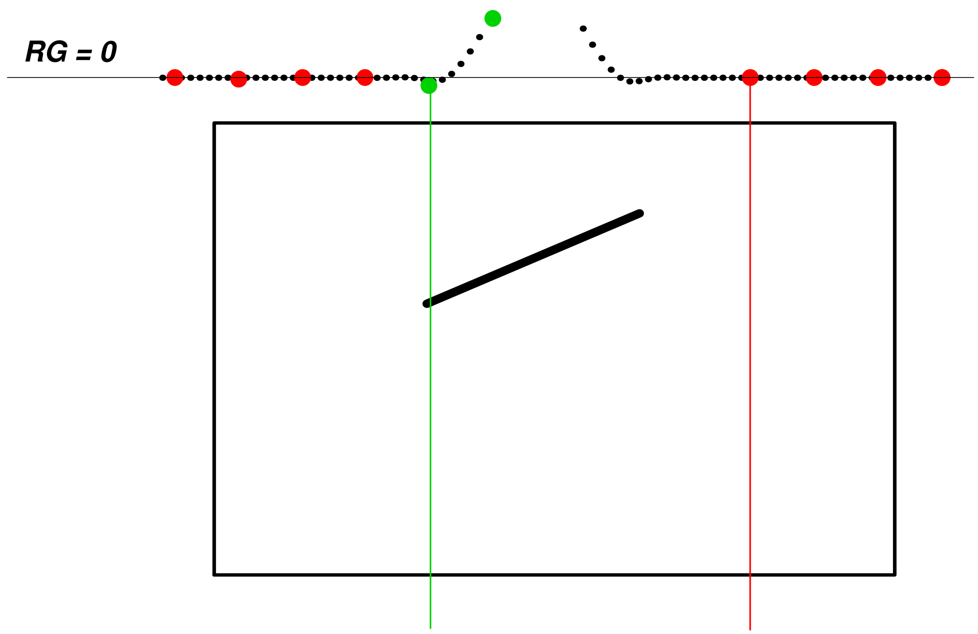


Réciprocité instantané

$$\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{q}H(t-\boldsymbol{x}\cdot\boldsymbol{p}) \qquad \boldsymbol{\sigma}[\boldsymbol{w}]\cdot\boldsymbol{n} = \boldsymbol{\tau}[\boldsymbol{w}]\delta(t-\boldsymbol{x}\cdot\boldsymbol{p})$$

$$\mathcal{R}(t) = \int_\Gamma [\![\boldsymbol{u}^t]\!]\cdot\boldsymbol{\tau}[\boldsymbol{w}]\,ds = \int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{\tau}[\boldsymbol{w}]\cdot\boldsymbol{n}\,ds$$

Ecart à la réciprocité: identification de la faille

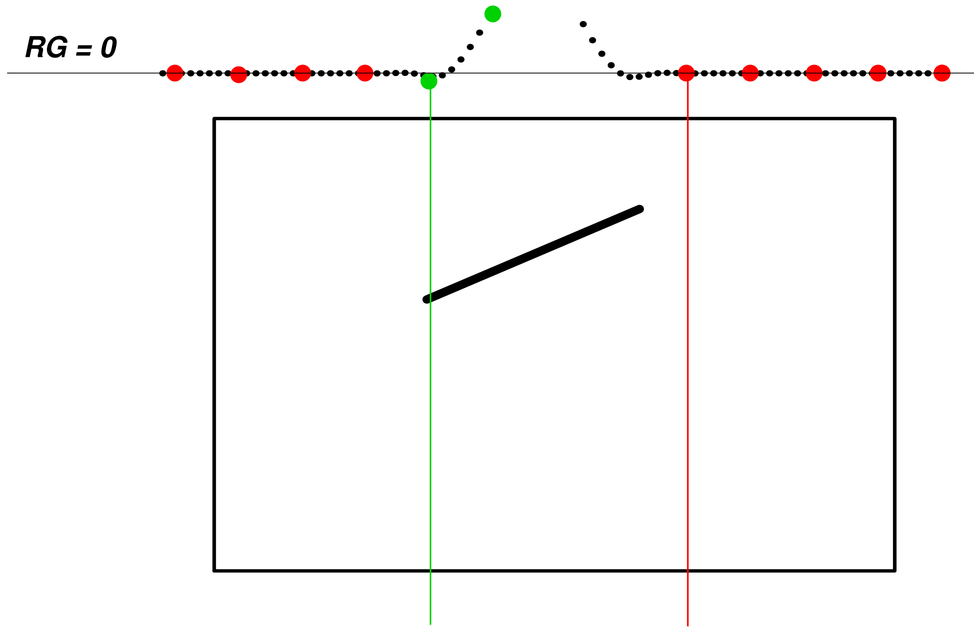


Réciprocité instantané

$$\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{q}H(t-\boldsymbol{x}\cdot\boldsymbol{p}) \qquad \boldsymbol{\sigma}[\boldsymbol{w}]\cdot\boldsymbol{n} = \boldsymbol{\tau}[\boldsymbol{w}]\delta(t-\boldsymbol{x}\cdot\boldsymbol{p})$$

$$\mathcal{R}(t) = \int_\Gamma [\![\boldsymbol{u}^t]\!]\cdot\boldsymbol{\tau}[\boldsymbol{w}]\,ds = \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{\tau}[\boldsymbol{w}]\cdot\boldsymbol{n}\,ds$$

Ecart à la réciprocité: identification de la faille

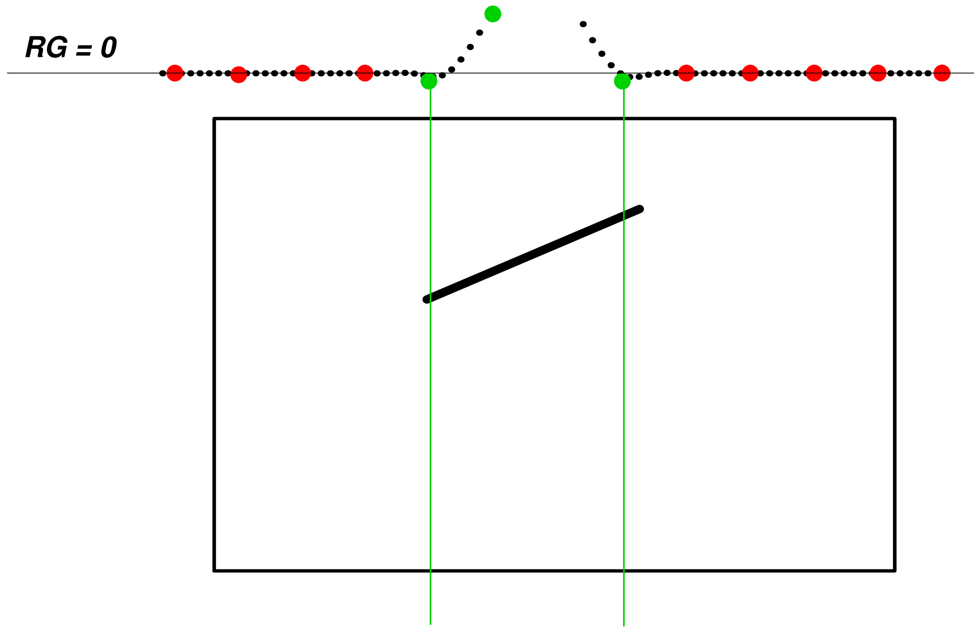


Réciprocité instantané

$$\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{q}H(t - \boldsymbol{x}\cdot\boldsymbol{p}) \qquad \boldsymbol{\sigma}[\boldsymbol{w}]\cdot\boldsymbol{n} = \boldsymbol{\tau}[\boldsymbol{w}]\delta(t - \boldsymbol{x}\cdot\boldsymbol{p})$$

$$\mathcal{R}(t) = \int_\Gamma [\![\boldsymbol{u}^t]\!]\cdot\boldsymbol{\tau}[\boldsymbol{w}]\,ds = \int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{\tau}[\boldsymbol{w}]\cdot\boldsymbol{n}\,ds$$

Ecart à la réciprocité: identification de la faille

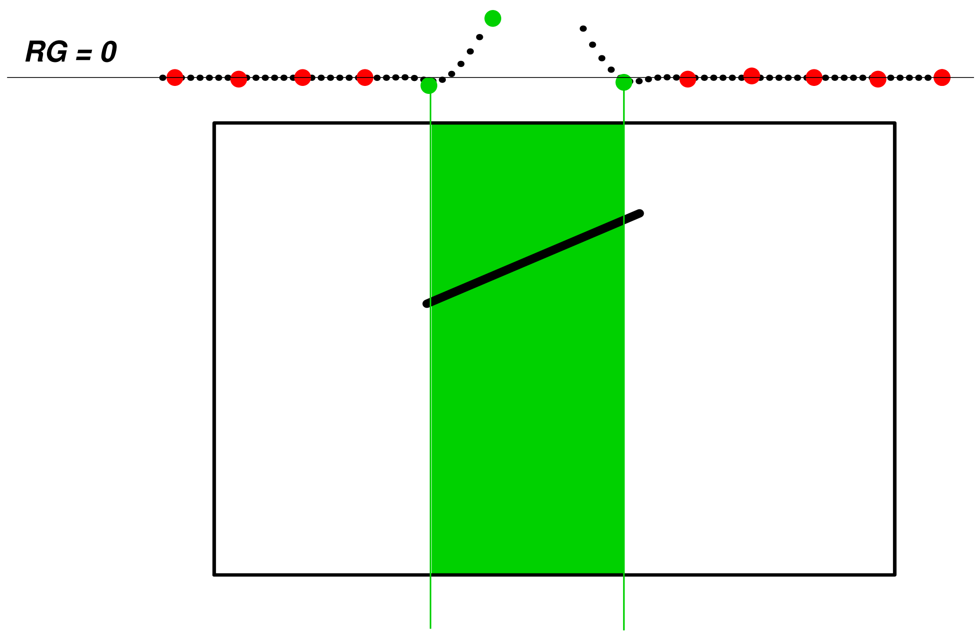


Réciprocité instantané

$$\boldsymbol{w}(\boldsymbol{x}, t) = \boldsymbol{q} H(t - \boldsymbol{x} \cdot \boldsymbol{p}) \qquad \boldsymbol{\sigma}[\boldsymbol{w}] \cdot \boldsymbol{n} = \boldsymbol{\tau}[\boldsymbol{w}] \delta(t - \boldsymbol{x} \cdot \boldsymbol{p})$$

$$\mathcal{R}(t) = \int_{\Gamma} [\![\boldsymbol{u}^t]\!] \cdot \boldsymbol{\tau}[\boldsymbol{w}] \, ds = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{\tau}[\boldsymbol{w}] \cdot \boldsymbol{n} \, ds$$

Ecart à la réciprocité: identification de la faille

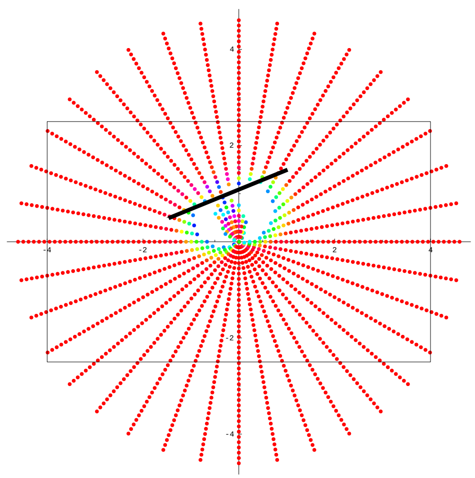


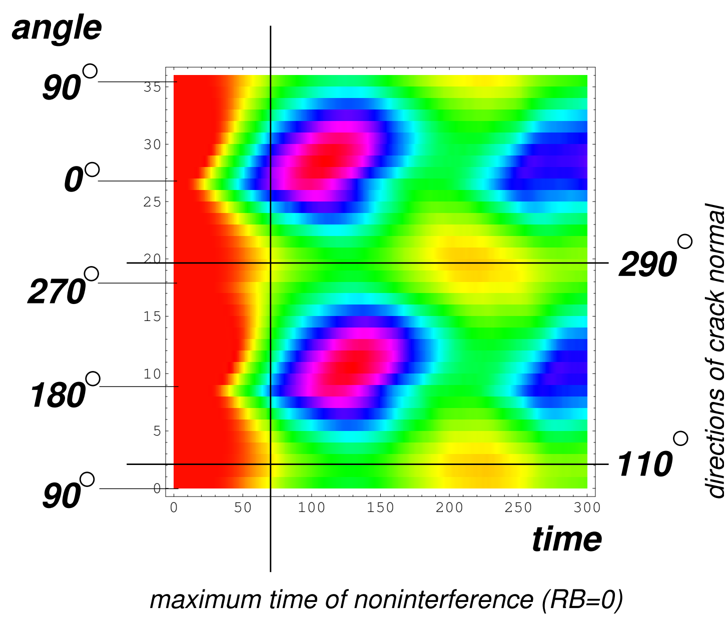
Réciprocité instantané

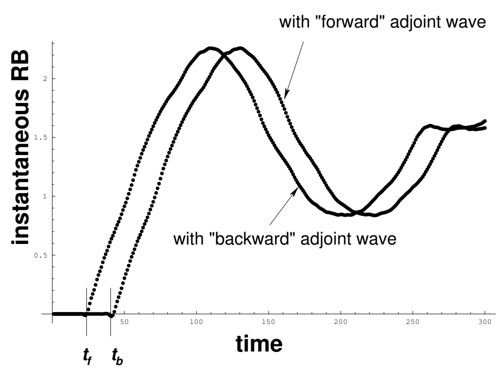
$$\boldsymbol{w}(\boldsymbol{x}, t) = \boldsymbol{q} H(t - \boldsymbol{x} \cdot \boldsymbol{p}) \qquad \boldsymbol{\sigma}[\boldsymbol{w}] \cdot \boldsymbol{n} = \boldsymbol{\tau}[\boldsymbol{w}] \delta(t - \boldsymbol{x} \cdot \boldsymbol{p})$$

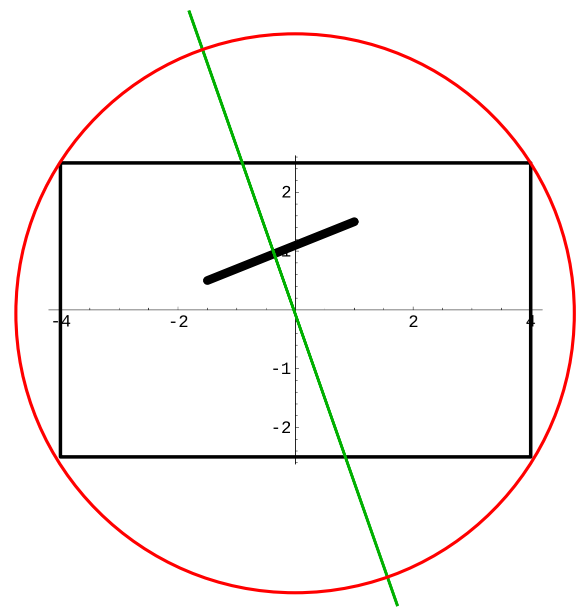
$$\mathcal{R}(t) = \int_{\Gamma} [\![\boldsymbol{u}^t]\!] \cdot \boldsymbol{\tau}[\boldsymbol{w}] \, ds = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{\tau}[\boldsymbol{w}] \cdot \boldsymbol{n} \, ds$$

Reciprocity - Crack Identification



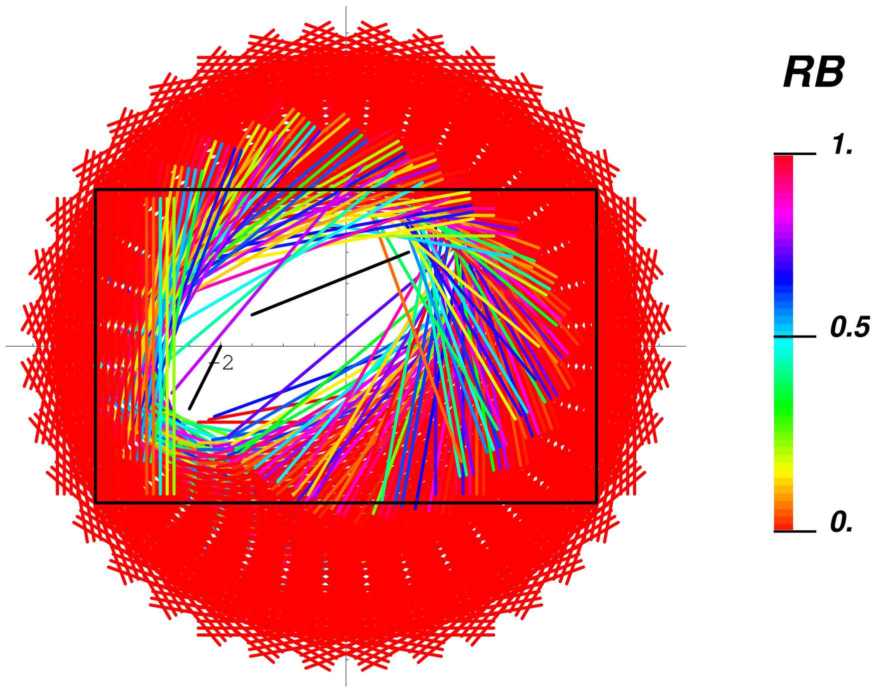






Reciprocity - Crack Identification

Convexe Hull



Error in constitutive law (ECL)

- electricity: Wexler & Mandel [1985], Kohn, Lowe, McKenney, Vogelius [1988-1990]
- elasticity: Ladèveze & Léguillon [1983] . . .

Balance of forces: $\quad \operatorname{div} \boldsymbol{C}^* \nabla \boldsymbol{u} = 0$

boundary conditions $\quad (S_u \cap S_p = \emptyset)$

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) &= \boldsymbol{u}^D(\boldsymbol{x},t) \quad (\boldsymbol{x},t) \in S_u \times [0,T] \\ \boldsymbol{\sigma}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) &= \boldsymbol{p}^D(\boldsymbol{x},t) \quad (\boldsymbol{x},t) \in S_p \times [0,T] \end{aligned}$$

Error on constitutive law

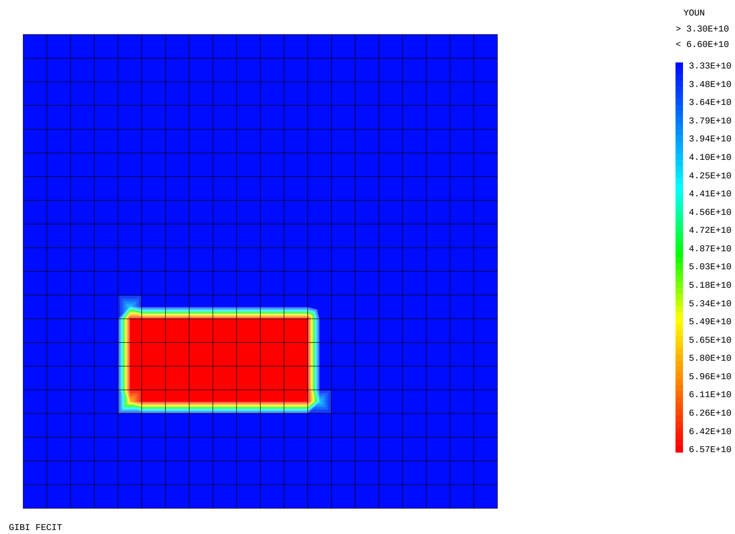
$$\begin{aligned} \mathcal{E}(\boldsymbol{v},\boldsymbol{s},\boldsymbol{C}) &= \mathcal{W}_C(\boldsymbol{v}) + \mathcal{W}^\star_C(\boldsymbol{s}) \\ &= \frac{1}{2}\int_\Omega \boldsymbol{\varepsilon}[\boldsymbol{v}]:\boldsymbol{C}:\boldsymbol{\varepsilon}[\boldsymbol{v}]\,dv + \frac{1}{2}\int_\Omega \boldsymbol{s}:\boldsymbol{C}^{-1}:\boldsymbol{s}\,dv - \int_{\partial\Omega} \boldsymbol{p}[\boldsymbol{n}]\cdot\boldsymbol{v}\,ds \\ & \quad \boldsymbol{s} \in \mathcal{S}(\boldsymbol{u}^D, S_u) \quad \text{et} \quad \boldsymbol{v} \in \mathcal{C}(\boldsymbol{p}^D, S_p) \\ E(\boldsymbol{C},\boldsymbol{v},\boldsymbol{s}) &= \frac{1}{2}\int_\Omega \|\boldsymbol{C}^{-1/2}:\boldsymbol{s} - \boldsymbol{C}^{1/2}:\boldsymbol{\varepsilon}[\boldsymbol{v}]\|^2\,dv \\ &= \frac{1}{2}\int_\Omega \Big(\boldsymbol{s} - \boldsymbol{C}:\boldsymbol{\varepsilon}[\boldsymbol{v}]\Big):\boldsymbol{C}^{-1}:\Big(\boldsymbol{s} - \boldsymbol{C}:\boldsymbol{\varepsilon}[\boldsymbol{v}]\Big)\,dv \end{aligned}$$

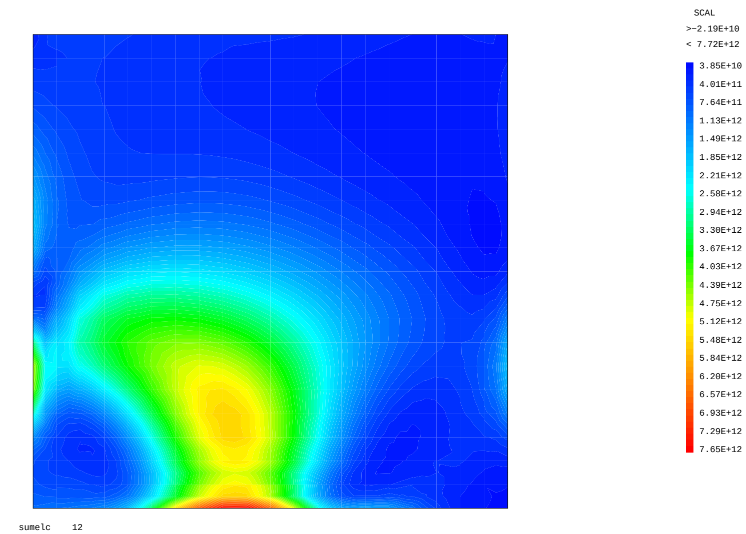
Fundamental Property

$$(\boldsymbol{u},\boldsymbol{\sigma}) \text{ solution} \quad \Leftrightarrow \quad E(\boldsymbol{C},\boldsymbol{u},\boldsymbol{\sigma}) = 0 \quad \Leftrightarrow \quad \boldsymbol{C} = \boldsymbol{C}^*$$

ECL - Localisation property

H.D.Bui & AC [2000]





Inclusion (right) and spatial distribution of ECL (left)

n=2,3

- $D = \text{supp}\,\delta \boldsymbol{C}$
- $d = \text{dist}(\boldsymbol{x}, D)$
- $\varepsilon[\boldsymbol{w}]^N, \varepsilon[\boldsymbol{w}]^D \approx d^-n$
- $\mathcal{E} \approx d^-2n$

ECL - Identification of $C^*(x)$

Balance of forces: $\operatorname{div} \boldsymbol{C}^* \nabla \boldsymbol{u} = 0$

boundary conditions

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^{D(i)}(\boldsymbol{x}) \quad (\boldsymbol{x}, t) \in S \qquad \boldsymbol{\sigma}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{p}^{D(i)}(\boldsymbol{x}) \qquad \boldsymbol{x} \in S \qquad (i = 1, m)$$

Minimisation ECL

$$\begin{aligned} \mathcal{E}(\boldsymbol{v}, \boldsymbol{s}, \boldsymbol{C}) &= \mathcal{W}_C(\boldsymbol{v}) + \mathcal{W}_C^\star(\boldsymbol{s}) \\ &= \frac{1}{2} \int_\Omega \boldsymbol{\varepsilon}[\boldsymbol{v}] : \boldsymbol{C} : \boldsymbol{\varepsilon}[\boldsymbol{v}] \, dv + \frac{1}{2} \int_\Omega \boldsymbol{s} : \boldsymbol{C}^{-1} : \boldsymbol{s} \, dv - \int_{\partial\Omega} \boldsymbol{p}[\boldsymbol{n}] \cdot \boldsymbol{v} \, ds \end{aligned}$$

Algorithm AC [1994,1995]

- initial distribution: $\boldsymbol{C}^{(0)}(\boldsymbol{x})$,
- compute $\boldsymbol{C}^{(i+1)}(\boldsymbol{x})$
  - compute

$$\boldsymbol{u}^{D(i)} \quad \text{et} \quad \boldsymbol{u}^{N(i)}$$

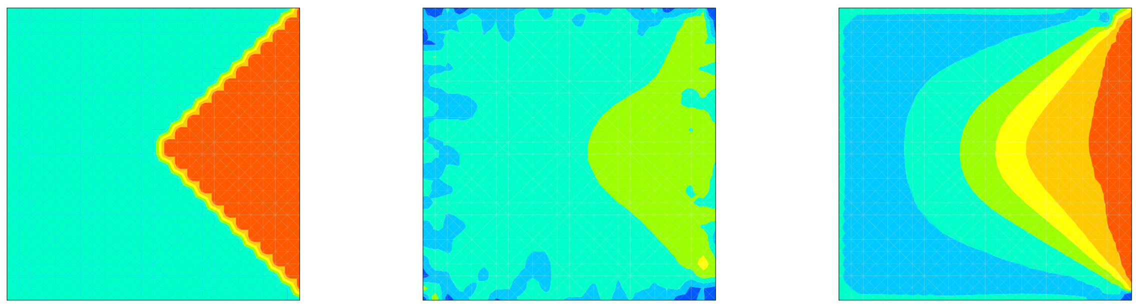
  - compute

$$\boldsymbol{C}_{i+1} = \arg\min_C E(\boldsymbol{C}, \boldsymbol{u}^{D(i)}, \boldsymbol{\sigma}^{N(i)})$$

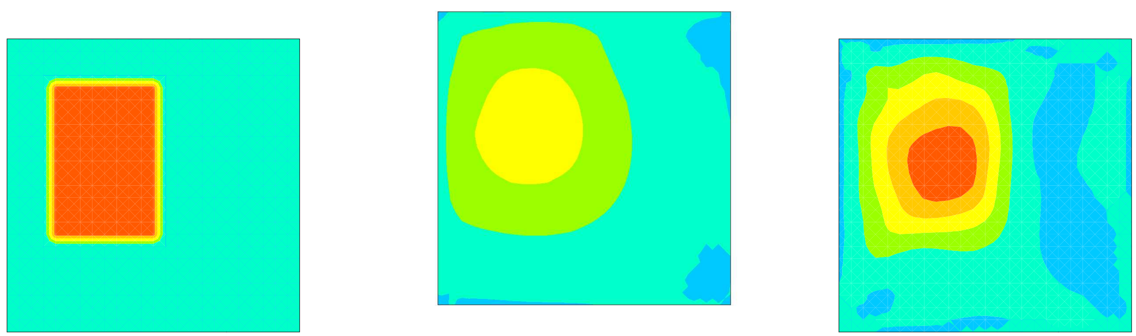
ECL - Identification of $C^*(x)$

Eigenelastic Moduli

$$C(\boldsymbol{x}) = \sum_{k=1}^{6} c_k(\boldsymbol{x})\, \boldsymbol{\xi}_k(\boldsymbol{x}) \otimes \boldsymbol{\xi}_k(\boldsymbol{x}) \qquad c_k^{(i+1)} = \left[\frac{\boldsymbol{\sigma}_k^{(i)} : \boldsymbol{\sigma}_k^{(i)}}{\boldsymbol{\varepsilon}_k^{(i)} : \boldsymbol{\varepsilon}_k^{(i)}} \right]^{1/2}$$



real | 2 iterations 10% noise | 14 iterations 0% noise



real | 5 iterations moments 0% noise | 32 iterations moments 0% noise

Identification of a square copper inclusion in an aluminium matrix
(Poisson's coefficient)

Inverse Problem

- **Inverse problems in elasticity**
  - General framework
  - Identification of cracks using the reciprocity gap
  - Identification of distributed elastic moduli
    error in constitutive law
- **Inverse problems and sensibility computations**
  - Sensibility and contact boundary conditions
  - Identification of parameters of constitutive laws:
    - ∗ Identation problem
    - ∗ Poroelasticity
- **Conclusion & Perspectives**

Nonlinear Problems (HPP)

Direct Problem $\mathcal{P}$ $\quad \boldsymbol{\sigma}(\boldsymbol{x},t) = \mathcal{F}\left(\varepsilon[\boldsymbol{u}], \varepsilon^p, \ldots\right)$

Balance of forces

$$\text{div}\,\boldsymbol{\sigma}[\boldsymbol{u}] + \boldsymbol{f} - \rho\,\ddot{\boldsymbol{u}} = 0$$

Initial Conditions $\quad (\boldsymbol{x} \in \Omega)$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \dot{\boldsymbol{u}}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x})$$

boundary conditions $\quad (\boldsymbol{x},t) \in S_u \cup S_p \times [0,T]$

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) &= \boldsymbol{u}^D(\boldsymbol{x},t) \\ \boldsymbol{\sigma}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) &= \boldsymbol{p}^D(\boldsymbol{x},t) \\ u_n - g - U^D(t) &\leq 0 \quad p \geq 0 \quad p(u_n - g - U^D) = 0 \end{aligned}$$

Inverse problems

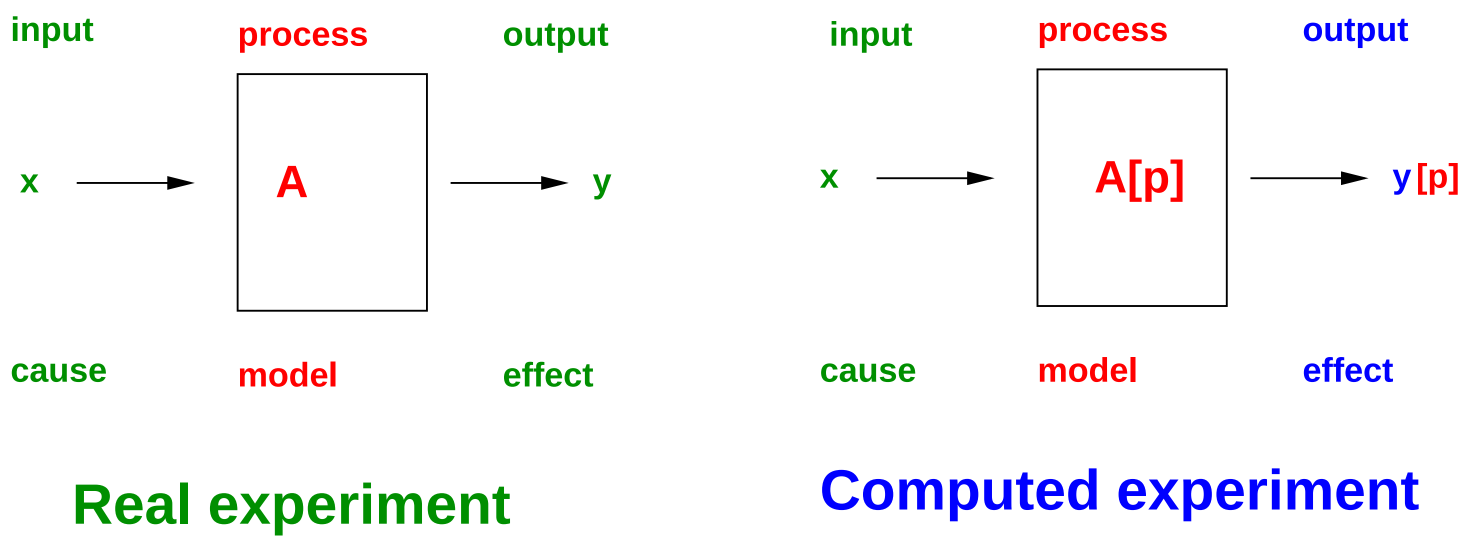
- viscoelasticity *[Chen et al., Moreau et al. CFM 2005], [Lorenzi at al.]*
- elastoviscoplasticity & contact *[Maier et al, AC & Tardieu]*
- process optimization *[Chenot et al., Zabaras et al., . . .]*

Technics

- Minimization of a cost functional:

$$p = \arg\min_{q,u\in\mathcal{P}q} \mathcal{J}(\boldsymbol{u},q)$$

Inverse Problem - Optimization Problem



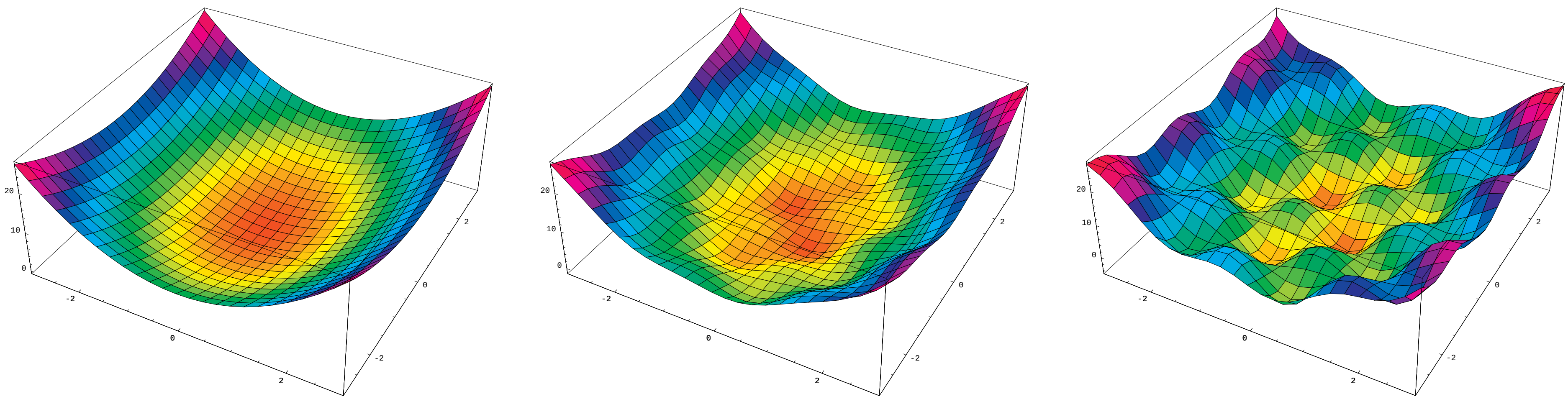
Problème d'optimization

$$p = \underset{q,u \in \mathcal{P}q}{\arg\min}\, \mathcal{J}(\boldsymbol{u}, q)$$

La fonctionnelle coût

$$\mathcal{J}(\mathbf{p}) = |y - y(\mathbf{p})|^2$$

La fonctionnelle coût $\mathcal{J}(\mathbf{p}) = |y - y(\mathbf{p})|^2$



Questions mathématiques

- convexité et propriétés de minimum
- erreures numeriques et expérimentales
- connaissances apriori sur les paramètres

Questions pratiques

- algorithm de minimisation: Gradient, Filtre Kalman, ...
- comment calculer

$$\frac{\partial \mathcal{J}}{\partial p} \quad \frac{\partial y}{\partial p}$$

  - Differences Finies
  - Differentiation Directe
  - Méthode de l'état adjoint

Sensitivities - Linear Direct Problem

Tortorelli & Michaleris, Vidal et. al., Arora et al., Kleiber et al, Zabaras et al.,

Minimize:

$$\mathcal{J}(\boldsymbol{p}) = \mathcal{J}(\boldsymbol{u}(\boldsymbol{p}), \boldsymbol{p})$$

with $\boldsymbol{u}(\boldsymbol{p})$ solution of:

$$\boldsymbol{K}(\boldsymbol{p})\boldsymbol{u}(\boldsymbol{p}) = \boldsymbol{F}(\boldsymbol{p})$$

$\boldsymbol{K}$ symetric positive definite - FEM stiffness matrix

- Finite Differences
- Direct Differentitiation
- Adjoint Method

Finite Differences

$$\mathcal{J}(\boldsymbol{p}+\Delta\boldsymbol{p}) = \mathcal{J}(\boldsymbol{p}) + \nabla\mathcal{J}(\boldsymbol{p})\cdot\Delta\boldsymbol{p} + \wr(\Delta\boldsymbol{p}^2) \qquad (1)$$

Forward Difference Approximation

$$\frac{d\mathcal{J}}{d\,p_i}(\boldsymbol{p}) = \frac{\mathcal{J}(\boldsymbol{p}+\Delta p_i) - \mathcal{J}(\boldsymbol{p})}{\Delta p_i} + \wr(\Delta p_i)$$

Backward Difference Approximation

$$\frac{d\mathcal{J}}{d\,p_i}(\boldsymbol{p}) = \frac{\mathcal{J}(\boldsymbol{p}-\Delta p_i) - \mathcal{J}(\boldsymbol{p})}{\Delta p_i} + \wr(\Delta p_i)$$

Centered Difference Approximation

$$\frac{d\mathcal{J}}{d\,p_i}(\boldsymbol{p}) = \frac{\mathcal{J}(\boldsymbol{p}+\Delta p_i) - \mathcal{J}(\boldsymbol{p}-\Delta p_i)}{2\Delta p_i} + \wr(\Delta p_i^2)$$

Advantage

- No preparation

Drawback

- Choice of Δp_i
- number of necessary computations $1+n_p, 1+2n_p$

Direct Differentiation

$$\mathcal{J}(\boldsymbol{p}) = \mathcal{J}(\boldsymbol{u}(\boldsymbol{p}), \boldsymbol{p})$$

$$\frac{d\,\mathcal{J}}{\partial p_i} = \frac{\partial \mathcal{J}}{\partial \boldsymbol{u}}(\boldsymbol{u}(\boldsymbol{p}), \boldsymbol{p})\frac{d\boldsymbol{u}}{dp_i} + \frac{\partial \mathcal{J}}{\partial p_i}(\boldsymbol{u}(\boldsymbol{p}), \boldsymbol{p})$$

$$\boldsymbol{K}\frac{d\boldsymbol{u}}{dp_i} = \frac{d\boldsymbol{F}}{dp_i} - \frac{d\boldsymbol{K}}{dp_i}\boldsymbol{u}$$

Advantage

- exact derivative
- FEM: same stiffness matrix

Drawback

- number of necessary computations $1 + n_p$
- same programming

Adjoint State Method Minimize $\mathcal{J}$ under constraint $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$

is equivalent to

Find the stationnarity point (saddle point) of the *Lagrangian*:

$$\mathcal{L}(\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{u}^*) = \mathcal{J}(\boldsymbol{u}, \boldsymbol{p}) - \boldsymbol{u}^* \left(\boldsymbol{K}(\boldsymbol{p})\boldsymbol{u} - \boldsymbol{F}(\boldsymbol{p})\right)$$

Stationnarity Conditions

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}^*} = \boldsymbol{K}(\boldsymbol{p})\boldsymbol{u} - \boldsymbol{F}(\boldsymbol{p}) = 0 \quad \text{Direct Problem} \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}} = -\boldsymbol{K}(\boldsymbol{p})^T \boldsymbol{u}^* + \frac{\partial J}{\partial \boldsymbol{u}} = 0 \quad \text{Adjoint Problem} \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{p}} = \frac{\partial J}{\partial \boldsymbol{p}} - \boldsymbol{u}^* \left(\frac{\partial \boldsymbol{K}}{\partial \boldsymbol{p}}\boldsymbol{u} - \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{p}}\right) \tag{4}$$

Advantage

- exact derivative
- FEM same stiffness matrix
- number of computations $1 + 1$

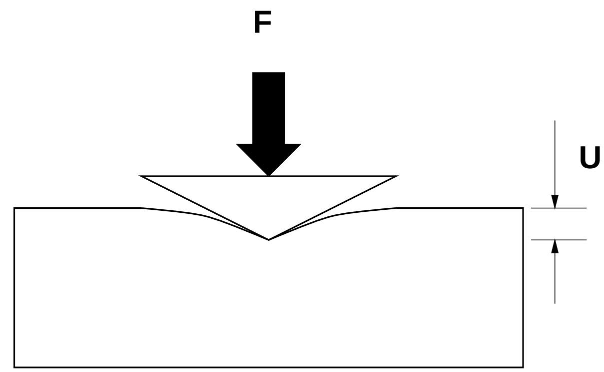
Sensitivity - elastoviscoplasticity

- Direct & Adjoint Differentation $\rightarrow \nabla\mathcal{J}$: `Cast3M` (CEA) or `Aster` (EDF)
- Gradient Descent - BFGS, Levenberg-Marquard, ... `Mathematica, Scilab`

Plan

- **Problèmes de minimization et calculs des sensibilités**
  - Problème d'optimization
  - Méthode des calculs des sensibilités
- **Problème d'indentation**
  - Calculs de sensibilité et conditions de contact
  - Essais d'indentation: Identification des paramètres de la loi de comportement
- **Poroélasticité**
  - Etat adjoint, Différentiation directe
  - Essai de compression drainée, Pulse test: Identification des paramètres de la loi de comportement
- **Conclusion et Perspectives**

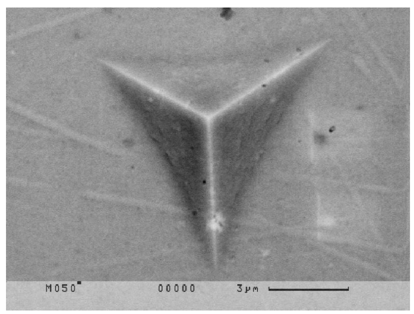
Problème d'indentation



Duréte

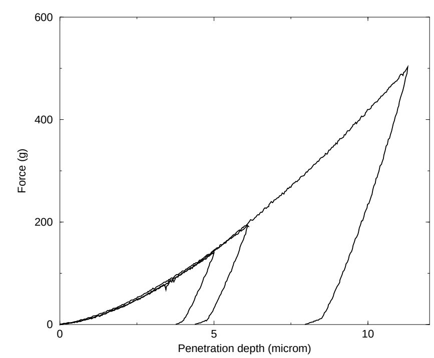
$$H_v \approx \frac{F}{Surface} = k\frac{F}{d^2} \approx H_0 d^{n-2}$$

Vickers



Berkovich

Indentation continue



Avantages: pas d'éprouvette, simple

Solutions

Formules exactes - demi espace

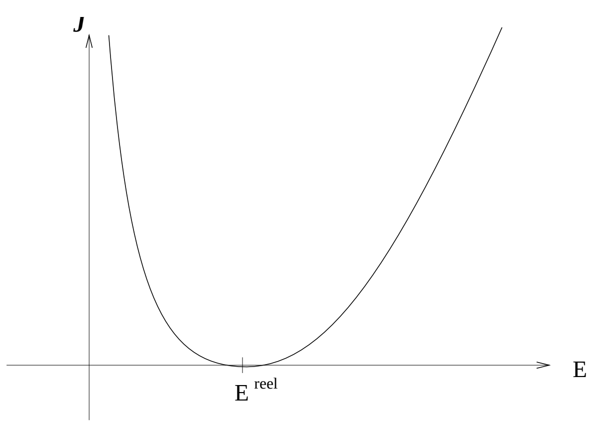
- Elasticity - Hertz [1882]
- Perfect Plasticity - Hill, Tabor [1954], . . .
- Plasticity with power laws - Jayaraman et al. [1988], . . .

Formules empiriques & approchées

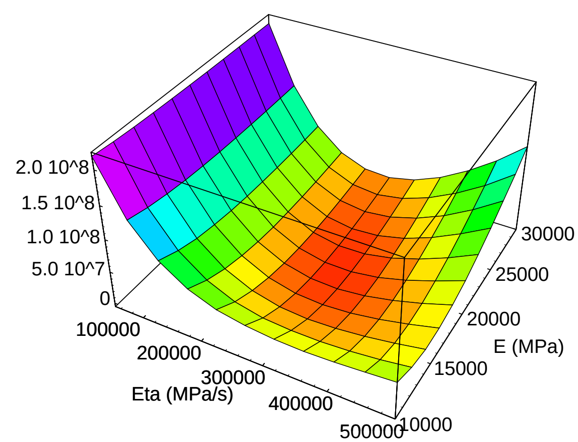
- films minces *[Hutchinson et al. 1998], [Korsunsky & AC 2000]*
- plasticité: sphère, cône *[Loubet et al., CFM 2005]*

Problème d'optimization $p = \arg\min_{q,u \in \mathcal{P}q} \mathcal{J}(\boldsymbol{u}, \Pi)$

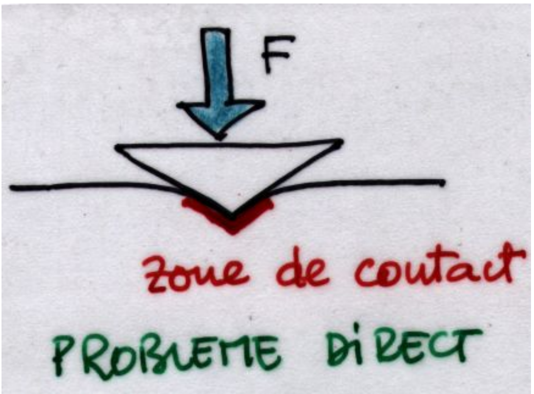
Elasticity



Maxwell viscous behaviour



Calculs de sensibilité et conditions de contact



Formulation primale ($\mathcal{P}$), Problème de Signorini

Trouver $\boldsymbol{u} \in \boldsymbol{K}$ *tel que*

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}) \, d\Omega \geq 0 \qquad (5)$$

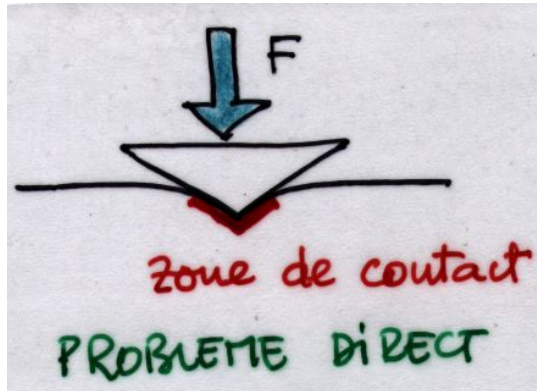
$$\forall \boldsymbol{v} \in \boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{V} \mid v_2 \leq g + U^{exp} \text{ on } \Gamma_C\}$$

Formulation mixte ($\mathcal{P}_m$)

Trouver $(\boldsymbol{u}, p) \in \boldsymbol{V} \times \boldsymbol{N}$ *tel que :*

$$\begin{aligned} &\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \, d\Omega - \int_{\Gamma_c} p \cdot v_2 \, d\Gamma = 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V} \\ &\int_{\Gamma_c} (q - p) \cdot (u_2 - U - g) \, d\Gamma \geq 0 \qquad \forall q \in \boldsymbol{N} \end{aligned} \qquad (6)$$

Calculs de sensibilité et conditions de contact



Problème d'optimization:

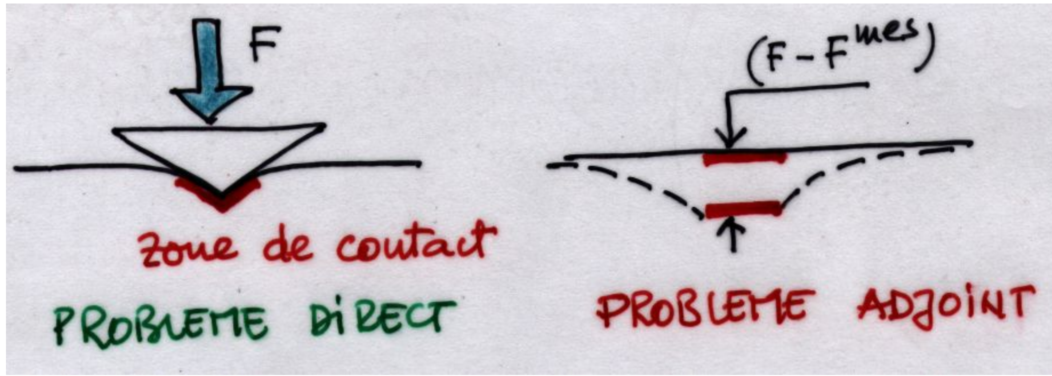
$$\boldsymbol{p} = \arg\min_{p \in \mathcal{A}} \mathcal{J}(\boldsymbol{p}, F^D) = \arg\min_{p \in \mathcal{A}, u \in solutions} \| F^D - F(\boldsymbol{p}, \boldsymbol{u}) \|$$

Lagrangien:

$$\begin{aligned} \mathcal{L}(\boldsymbol{u}, \boldsymbol{v}, p, q, \boldsymbol{c}) &= \frac{1}{2}(F - F^D)^2 - \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\, d\Omega \\ &+ \int_{\Gamma_C} p \cdot v_2\, d\Gamma + \int_{\Gamma_C} q \cdot (u_2 - U - g)\, d\Gamma \end{aligned}$$

Point d'optimalité = stationnarité de lagrangien

Calculs de sensibilité et conditions de contact



Adjoint Problem: *Trouver* $\boldsymbol{v} \in \boldsymbol{V}^{adj}$ *tel que :*

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{w})\, d\Omega &= 0 \\ \forall \boldsymbol{w} \in \boldsymbol{V}^{adj} &= \{\boldsymbol{w} \in \boldsymbol{V} \mid w_2 = (F - F^D) \text{ on } \Gamma_C^{eff}\} \end{aligned}$$

Gradient de $\mathcal{J}$:

$$[\frac{\partial \mathcal{J}}{\partial \boldsymbol{p}}, \boldsymbol{q} - \boldsymbol{p}] = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \frac{\partial \boldsymbol{C}}{\partial \boldsymbol{p}} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot (\boldsymbol{q} - \boldsymbol{p})\, d\Omega \geq 0 \qquad \forall \boldsymbol{q} \in \mathcal{A}$$

Remarques

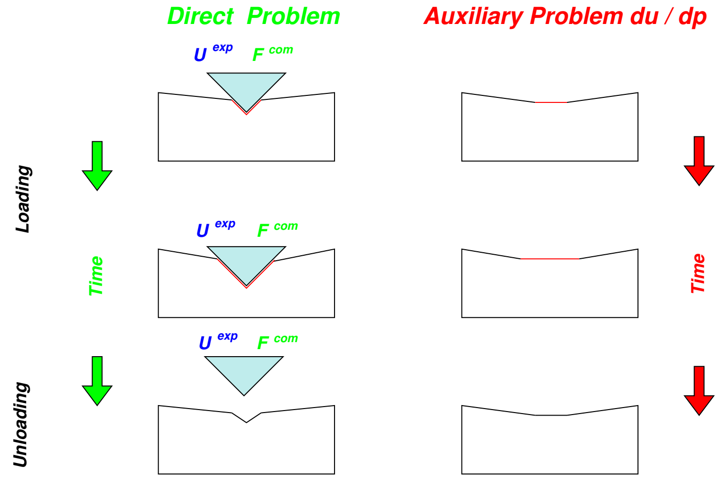
- différence entre continu et $\mathbb{R}^n$
- Mignot, Bergounioux, Kunish [1987-2002]

Calculs de sensibilité et problèmes d'évolution

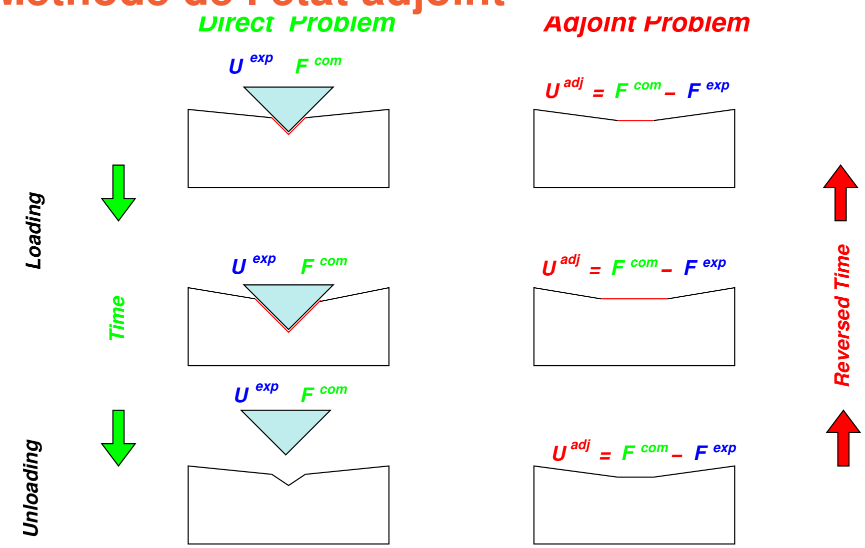
Direct Problem $\varepsilon(\Delta \boldsymbol{u}_i) = \boldsymbol{S} : \Delta\boldsymbol{\sigma}_i + \frac{\partial\Phi(\boldsymbol{\sigma}_i, \boldsymbol{c})}{\partial\boldsymbol{\sigma}}\Delta t$

Problème d'optimzation $\boldsymbol{p} = \text{arg min}_{p\in\mathcal{A}}\mathcal{J}(\boldsymbol{p}, \boldsymbol{D})$

Différentiation directe



Méthode de l'état adjoint



$$\boldsymbol{K}_i D_p \boldsymbol{u}_i = -\partial_p \boldsymbol{K}_i \boldsymbol{u}_i + \partial_p \boldsymbol{f}_i$$

Matrice tangente coherente *[Simo, Vidal, Arrora, . . .]*

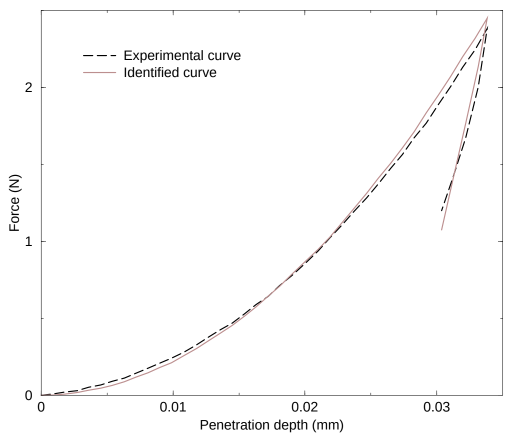
$$\varepsilon(\Delta \boldsymbol{u}_i^\star) = \boldsymbol{S} : \Delta\boldsymbol{\sigma}_i^\star - \frac{\partial^2\Phi(\boldsymbol{\sigma}_i, \boldsymbol{c})}{\partial\boldsymbol{\sigma}^2}\Delta t : \boldsymbol{\sigma}_i^\star$$

$$\nabla_{\boldsymbol{p}}\mathcal{J} = \sum_{i=0}^{I}\left(\int_\Omega \Delta\boldsymbol{\sigma}_i : \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{p}} : \boldsymbol{\sigma}_i^\star + \frac{\partial^2\Phi}{\partial\boldsymbol{\sigma}\partial\boldsymbol{p}}\Delta t : \boldsymbol{\sigma}_i^\star d\Omega\right)$$

Exemple: problème d'indentation

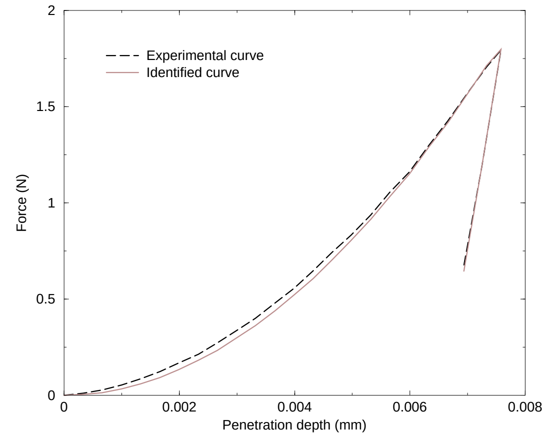
Norton - Hoff: AC & N.Tardieu [2000]

Nylon



	Initial Values	Final Values Iteration 56
E (MPa)	1000.	1930.
K (MPa.s$^{1/n}$)	100.	83.95
n	4.	7.34
σ^y (MPa)	30.	15.59
$\mathcal{J}$	2.08	0.020

Duraluminium

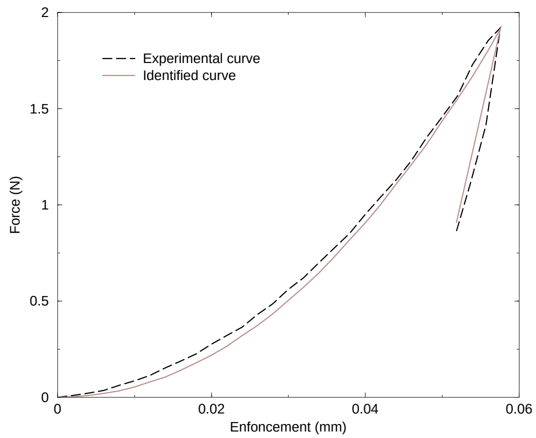


	Initial Values	Final Values Iteration 43
E (MPa)	50000.	37471.8
K (MPa.s$^{1/n}$)	1500.	2750.76
n	5.	4.60
σ^y (MPa)	100.	52.09
$\mathcal{J}$	0.22	0.0034

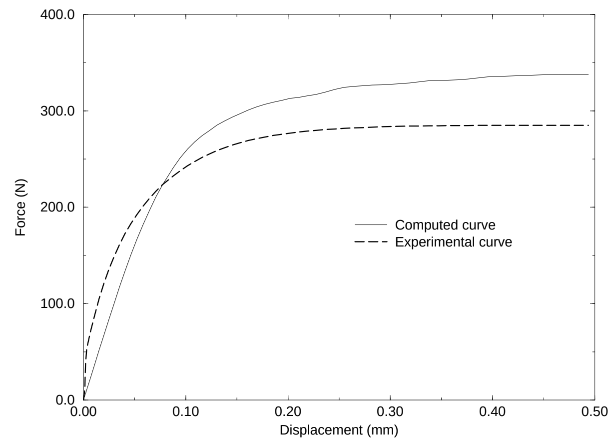
Exemple: problème d'indentation

Norton - Hoff: AC & N.Tardieu [2000]

Identation

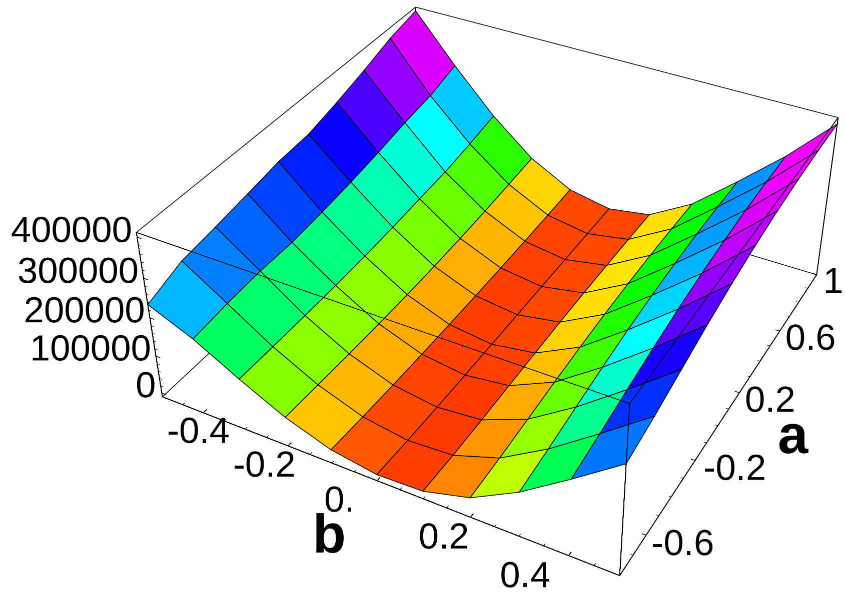


Traction



	Initial Values	Final Values Iteration 53
E (MPa)	200.	520.
K (MPa.s$^{1/n}$)	10.	7.90
n	4.	7.93
σ^Y (MPa)	6.	9.28
$\mathcal{J}$	3.85	0.039

Identifiabilité ?



Questions sur l'identification des paramètres

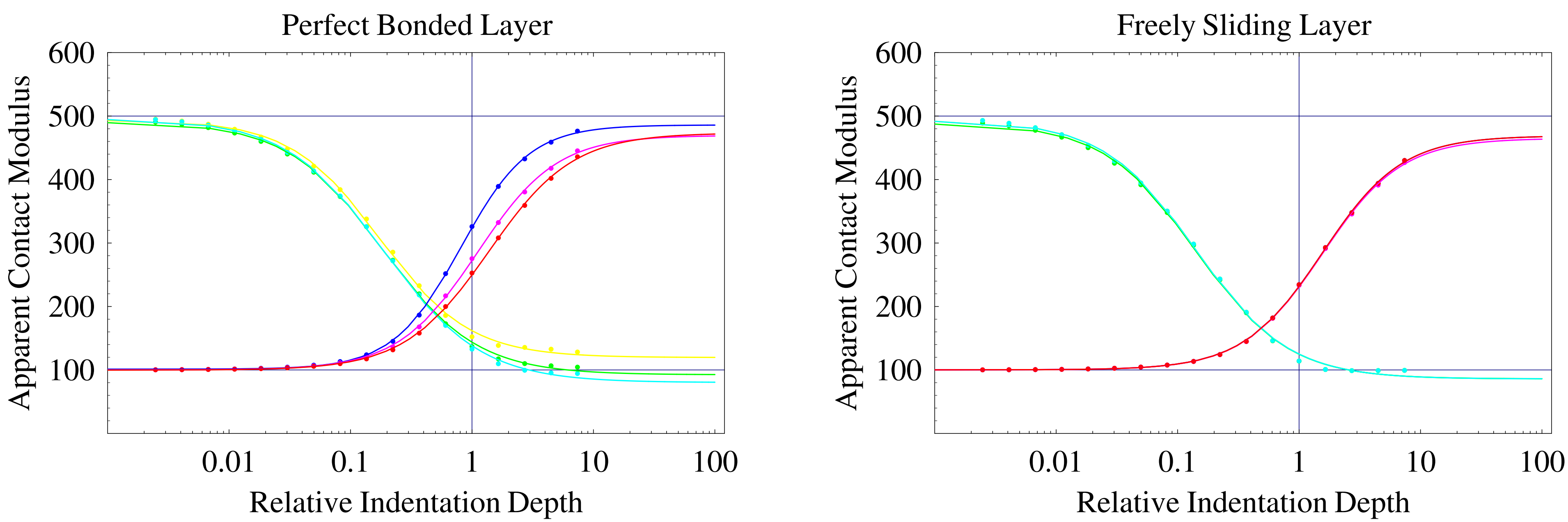
- existence et unicité
- stabilité (type de continuité)

Identation et couches minces

Module apparent

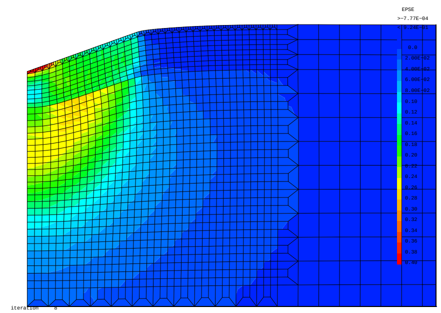
- couches élastiques, Indenteur cône
- Comparaison calcul semianalytiques - Prédiction formule:

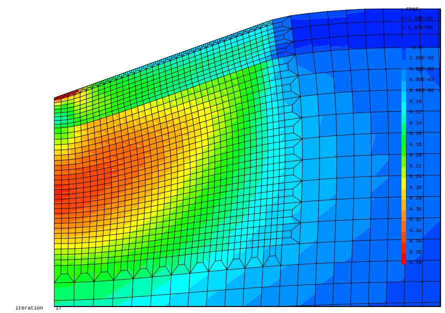
$$E^*(d) = e_2^* + \frac{e_1^* - e_2^*}{1 + \left(\frac{1}{\beta_0}\frac{d}{h}\right)^{\eta}} \qquad (7)$$

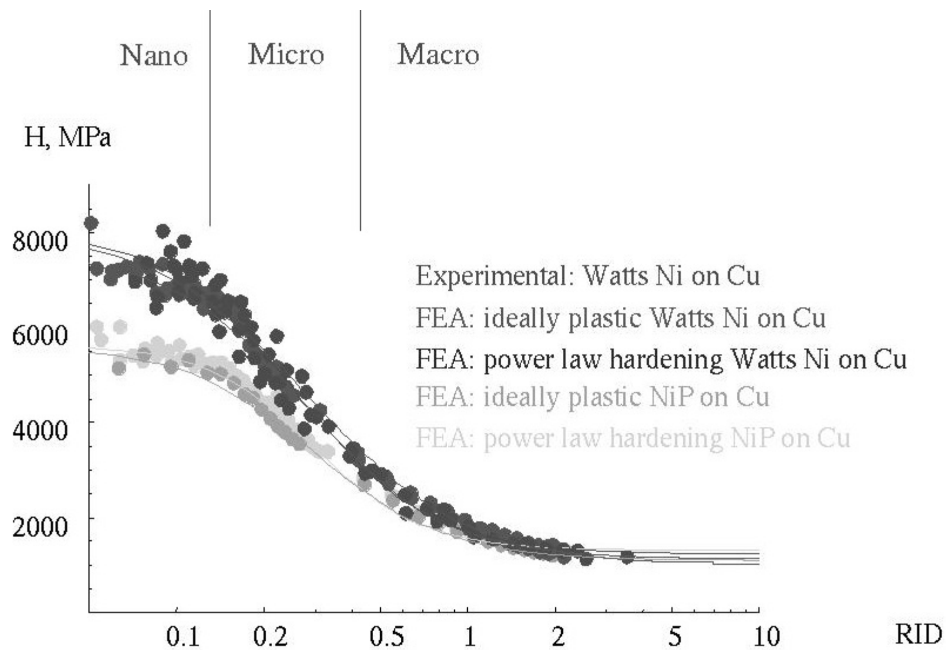


Identation et couches minces Calcul EF

- couches élastoplastiques, Indenteur cône
- Comparaison calcul semianalytiques - Prédiction formule - Expériments:







Plan

- **Problèmes de minimization et calculs des sensibilités**
  - Problème d'optimization
  - Méthode des calculs des sensibilités
- **Problème d'indentation**
  - Calculs de sensibilité et conditions de contact
  - Essais d'indentation: Identification des paramètres de la loi de comportement
- **Poroélasticité**
  - Etat adjoint, Différentiation directe
  - Essai de compression drainée, Pulse test: Identification des paramètres de la loi de comportement
- **Conclusion et Perspectives**

Poroélasticité

Direct Problem $\mathcal{P}$

EDP paraboliques [Biot 1941,....]:

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{F} = \mathbf{0}$$

$$p$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$$

$$\dot{\zeta}(\boldsymbol{q}) = -\operatorname{div} \boldsymbol{q} + g$$

Hydraulic diffusion equation

$$\boldsymbol{q} = -\boldsymbol{k} \cdot (\nabla p - \boldsymbol{f})$$

Mechanic constitutive law

$$\boldsymbol{\sigma} - \boldsymbol{\sigma}_0 = \boldsymbol{C}_u : \boldsymbol{\varepsilon}(\boldsymbol{u}) - M\boldsymbol{b}\zeta(\boldsymbol{q})$$

$$p - p_0 = M(\zeta(\boldsymbol{q}) - \boldsymbol{b} : \boldsymbol{\varepsilon}(\boldsymbol{u}))$$

++Conditions initiales et au limites

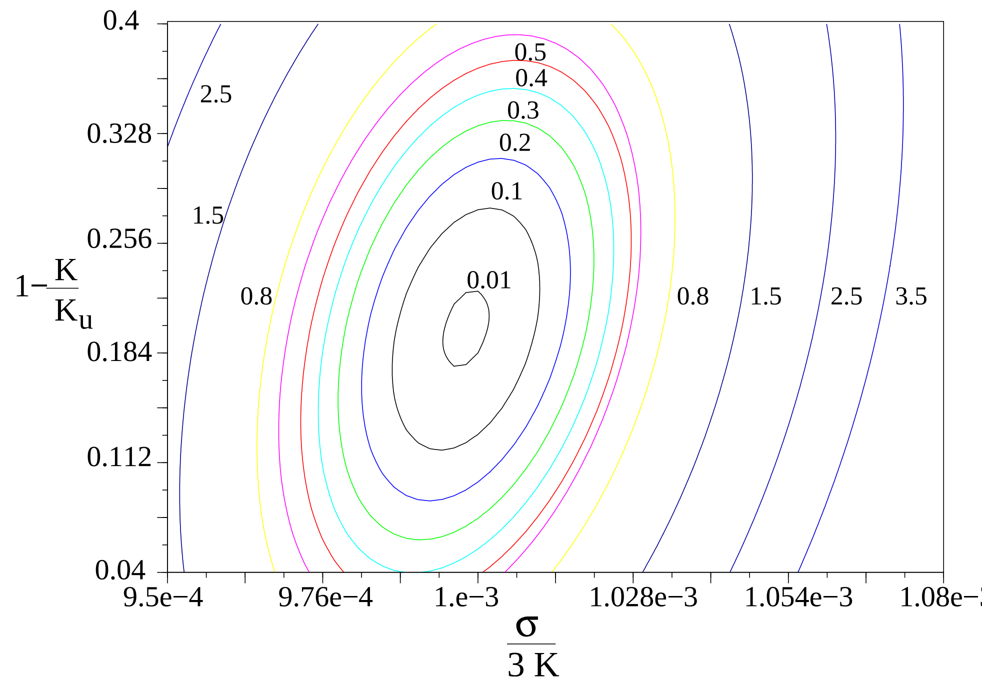
Problème inverse $\mathcal{P}^{-1}$

Déterminer $\boldsymbol{p} = \{\boldsymbol{k}, \boldsymbol{C}_u, \boldsymbol{b}, M\}$ des mesures.

Différentiation directe et méthode de l'état adjoint

Lecampion & AC [IJNAMG 2005, IJG 2005]

Méthode d'identification



Minimisation d'une fonctionnelle coût :

$$p = \underset{q,u \in \mathcal{P}q}{\text{arg min}}\, \mathcal{J}(\boldsymbol{u}, q)$$

Unicité, Stabilité de l'application:
$p \longrightarrow \mathcal{J}(\boldsymbol{c})$

Confiment isotrope drainée

$$\boldsymbol{c} = \left(K,\ K_u,\ D = kM\frac{K}{K_u} \right)$$

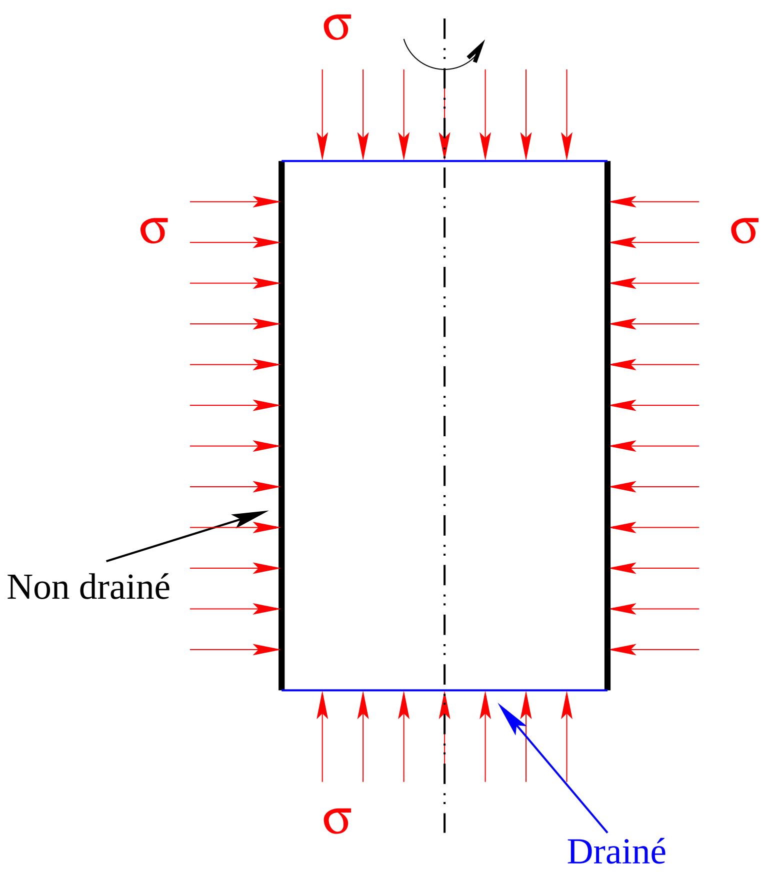
Pulse Test :

$$\boldsymbol{c} = \left(D,\ \gamma = \frac{MK}{C_r K_u \pi R^2 L} \right)$$

Technique: *Différentiation Directe* + minimization avec Levenberg Marquardt

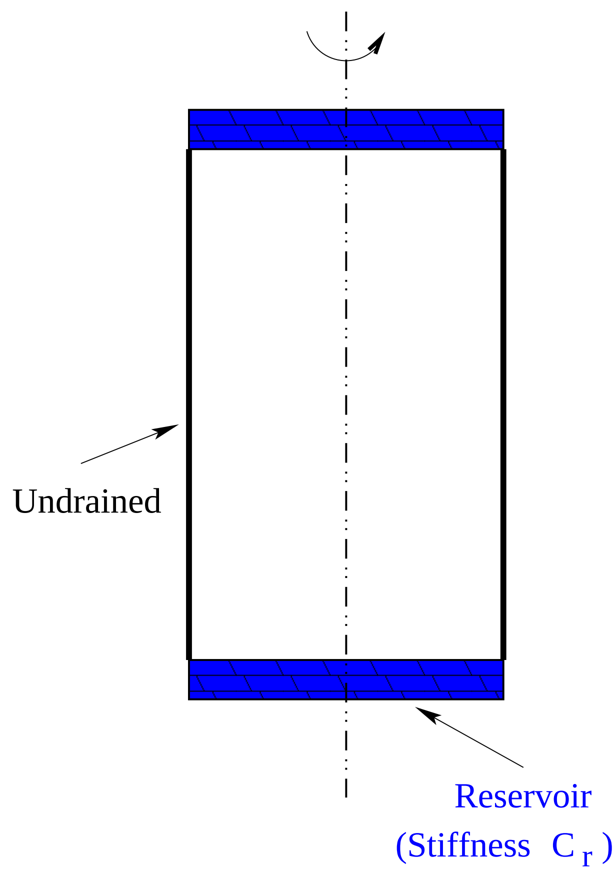
Exemple poroélasticité

Compression isotrope drainée



- pression de confinement à $t = 0$
- Mesure désplacements :
$u_z(L, t)$ & $u_r(L/2, t)$

Pulse test



- pas de changements en pression
- At $t = 0$: pressure fluide injecté $p_r = p_0$
- Mesure pression réservoire : $p_r(t)$

Exemple poroélasticité

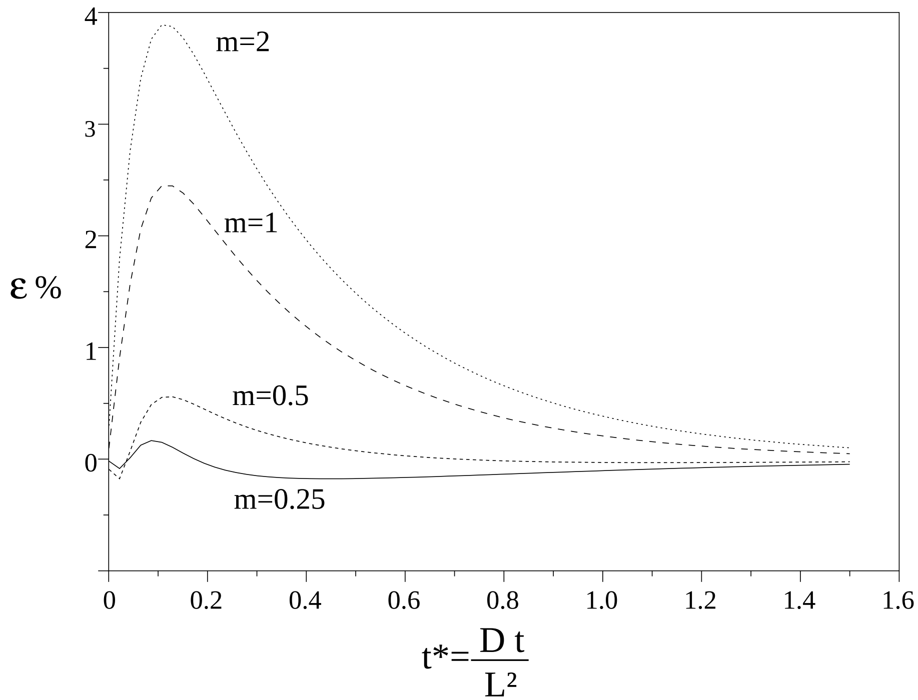
Rapport d'aspect $m = R/L \ll 1 \longrightarrow$ solution semi-analytique

Pulse test : solution semi-analytique: Hsieh et al. [1981] (transformé de Laplace)

compression drainée :solution semi-analytique (série)

integration numérique $\longrightarrow$ *Mathematica*

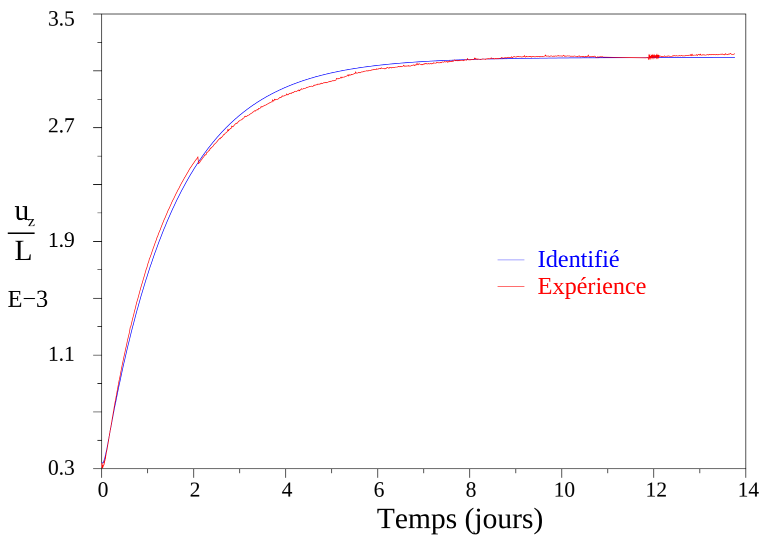
Comparison with 2D-FEM solutions (`Cast3M`)



Exemple poroélasticité

Compression isotrope drainée

déplacement axial



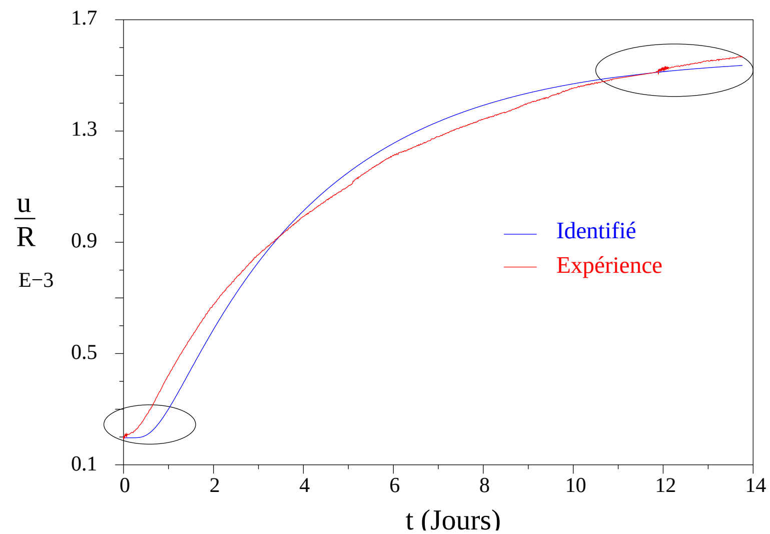
$K = 1.04GPa,\ K_u = 9.82GPa,$
$D = 4.02\,10^{-9}m^2.s^{-1}$

$\Rightarrow$ estimation de la perméabilité intrinséque

$$D = \frac{\kappa}{\mu_f} M \frac{K}{K_u})$$

$M\,(GPa)$	$\kappa\,(m^2)$
10	$3.7\,10^{-21}$
15	$2.5\,10^{-21}$

déplacement radial



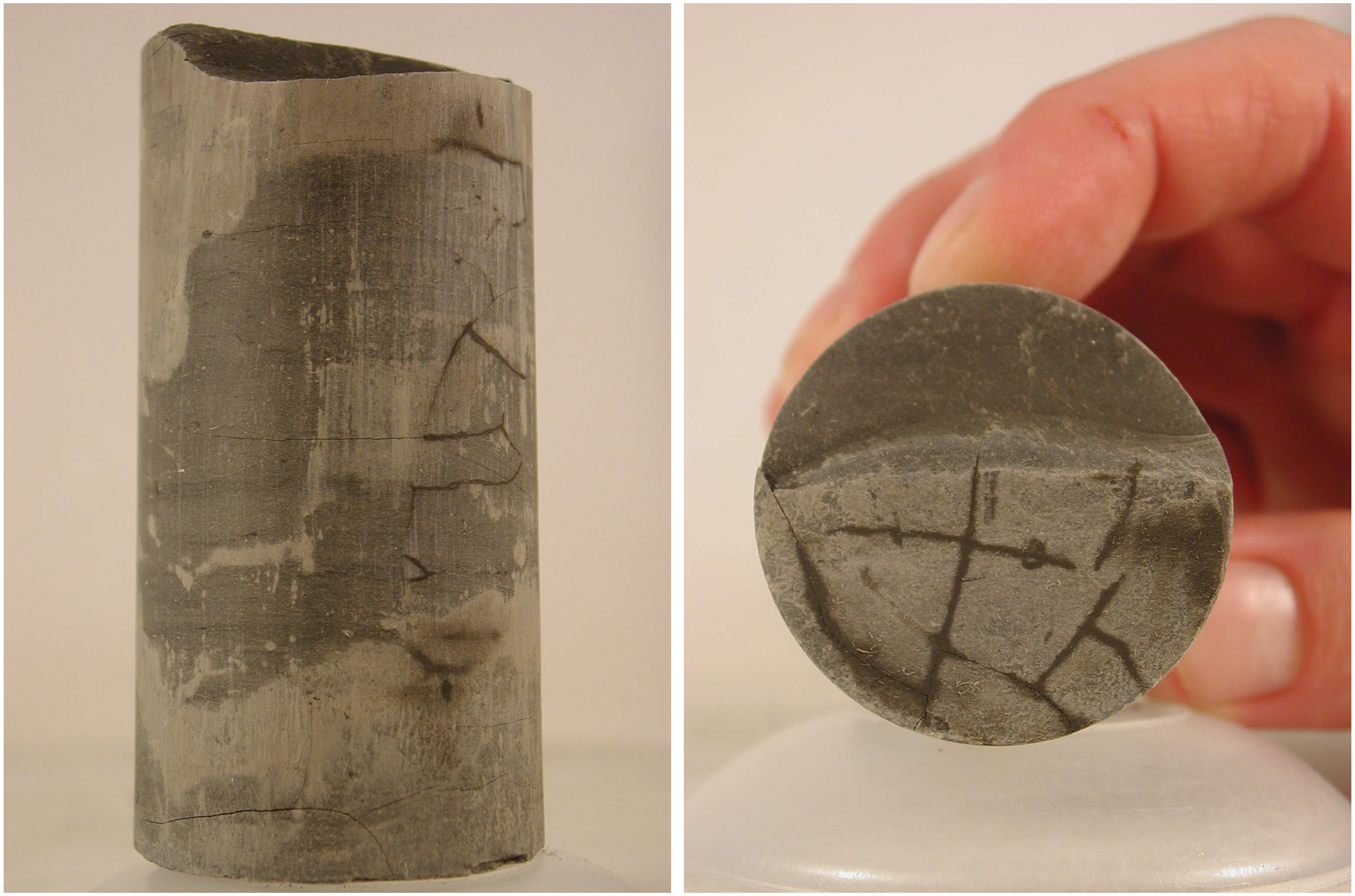
$K = 2.12GPa,\ K_u = 16.9GPa,$
$D = 1.74\,10^{-9}m^2.s^{-1}$

$\Rightarrow$ Anisotropie, Fissuration

Corrélation : $C_{KD} = 0.85$

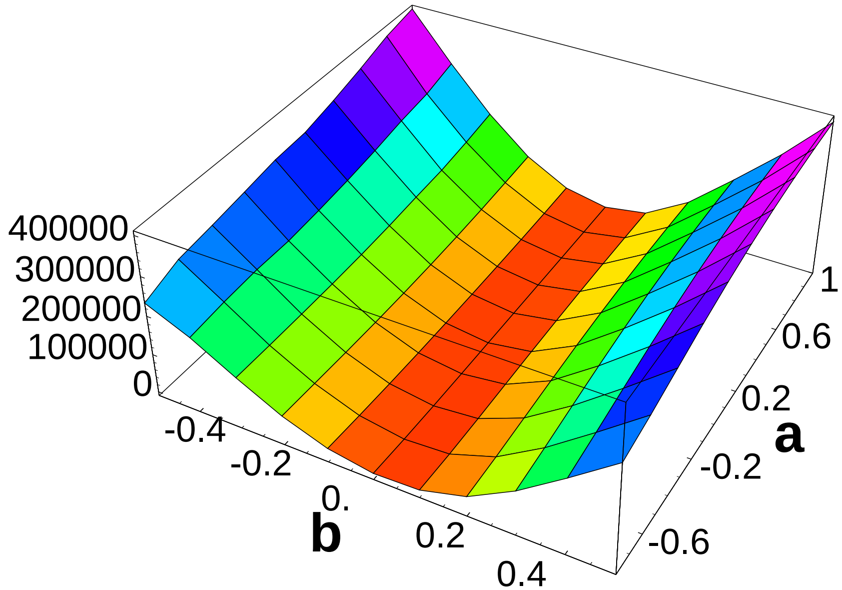
$\Rightarrow$ autres phénomènes : réactions chimiques

Exemple poroélasticité



Les éprouvettes (LMS-G3S)

Identifiabilité ?



Questions sur l'identification des paramètres

- existence et unicité
- stabilité (type de continuité)

Conclusion

Identification in elasticity

- theoretical and numerical results
- direct differentiation & adjoint state method

Sensibility Computations

- sensibility and contact conditions
- direct differentiation & adjoint state method

Perspectives

Techniques

- automatisation
- new applications

Theory

- generalization for constitutive laws with a sous-differentiable potential
- friction contact ?
- uniqueness & stability questions
- statistical analysis & correlation
- model pertinence