Which first passage problems are solvable?

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Contents

1 Markovian semigroups and infinitesimal generators

Jump-diffusions. A recurring theme in applied probability is distinguishing between two possible sources of uncertainty: small continuous changes modeled by diffusion, and "catastrophic" changes modeled by a jump process. To resolve this issue, one uses jump-diffusions models, i.e. solutions of a SDE (stochastic differential equation)

$$dX_t = \varphi(X_t)dt + \sigma(X_t)dB_t - dS_t \qquad (1)$$

where

- B_t is standard Brownian motion,
- S_t is a pure jump process, with a "Levy density" $\nu(x,z) := \lambda(x)\, b(z)$, which may arise for example from i.i.d. jumps C_i (sometimes one sided, for example negative), whose density, distribution, complementary distribution and first moment are denoted respectively by $b(x), B(x), \bar{B}(x), b_1$.

The first two terms (the drift φ and the variance σ) of the Levy-Khinchine triple φ, σ, ν define a continuous diffusion process, and the last term defines a pure jump/convolution process.

Jump-diffusions (1) give rise to **Markovian semigroups** of transition operators with associated evolution/backward Kolmogorov equation

$$\frac{\partial f(x,t)}{\partial t} = \mathcal{G}_x f(x,t), \quad f(x,0) = f_0(x), \tag{2}$$

which describes "expectations evolving in time"

$$f(t,x) = \mathbb{E}_{X_0=x} f_0(X_t).$$

The infinitesimal generator [§] operator is given by

$$\begin{aligned} \mathcal{G}f(x) &= \mathcal{G}_x f(x) = \varphi(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x) + \quad (3)\\ &\int_{-\infty}^{\infty} (f(x-z) - f(x))\nu(x,z)dz \\ &= \mathcal{G}^{(d)}f(x) + \mathcal{G}^{(j)}f(x), \quad (4) \end{aligned}$$

for any twice continuously differentiable and bounded function $f(x)$, where the second part $\mathcal{G}_x^{(j)}$ is associated to the pure jump convolution part.

Remark 1. *Note that in the simplest case of random walks, i.e. Markovian processes with discrete state spaces, the semigroups are simply matrix exponentials.*

Lie theory would seem to be "taylor made" for dealing with the more complicated jump-diffusion processes

[§]This term seems to originate with Lie.

(1), *whose generators* (3) *combine generically non-commuting operators.*

Indeed, while the operator semigroup may still be written formally as $\mathrm{e}^{t(\mathcal{G}^{(d)}+\mathcal{G}^{(j)})}$, *computing it in terms of the two individual semigroups becomes more complicated.*

One important example of diffusions, already studied in Kolmogorov's founding paper [57], is that of **hypergeometric diffusions** with quadratic variance and linear drift:

$$\mathcal{G}^{(d)} f(x) := \left(a_2 x^2 + a_1 x + a_0\right) \frac{\partial^2 f}{\partial x^2} + (\varphi_1 x + \varphi_0) \frac{\partial f}{\partial x}.$$

The Levy model is obtained when the variance and drift rates as well as the Levy intensity $\nu(x, z)$ are independent of x.

Example 1. The Cramér Lundberg risk model *(1903) [71] , one of the most studied models in applied probability, describes the surplus of an insurance company:*

$$U(t) = u + c\,t - S(t) := u + c\,t - \sum_{i=1}^{N(t)} C_i \qquad (5)$$

with initial capita u, *linear premium rate/drift* $c\,t$ *and "claims"* C_k, *modeled by a sequence of i.i.d. positive random variables with a common density* $f(x) =$

$f_C(x)$, and which arrive at the increase points $N = \{N_t, t \geq 0\}$ of an independent counting process with $\mathbb{E}N_t = \lambda t$. If moreover N_t is a Poisson process with exponential interarrival times, than $S(t)$ is a compound Poisson process with positive summands, and $U(t), t \in \mathbb{R}_+$ is Markovian.

2 The importance of Lie algebras in analysis

The relevance of finitely generated Lie algebras for solving differential systems was discovered by Lie, who established the equivalence of **superposition principles** for first order nonautonomous systems to that of a Lie algebra so that $G_x \in= \mathfrak{g}$. Furthermore, other algebra features play a role: for example the solvability of the Lie algebra implies integrability by quadratures of the system in the sense of Liouville – see [**?**] .

The first question of interest for Markovian semigroups/Lie groups is the explicit computation of the transition operators. Similarly with with Lie's celebrated superposition theorem, this is possible precisely when the infinitesimal generator satisfies that

$$G_x \in= \mathfrak{g} = \{\sum_{i=1}^{I} c_i G^{(i)}\}$$

for some Lie algebra $\mathfrak{g}$.

Then, one may compute the exponential of each generator $G^{(i)}G^{(i)}$ separately, and then combine them via formulas like Baker-Campbell-Haussdorff-Dynkin or Wei-Norman's.

Informately, we must be able to break the infinitesimal generator as a linear combination of a finite number of terms, which give rise to a finitely generated Lie algebra under commutation and scalar multiplication.

Example 2. *Consider the elementary example of Brownian motion with drift with associated generator*

$$\mathcal{G}_x f = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} + c\frac{\partial f}{\partial x}.$$

Since the two operators D and D^2 **commute**, *the resulting exponential $e^{t\mathcal{G}}f(x)$ may be decomposed as*

$$e^{t\mathcal{G}}f(x) = e^{tcD}e^{t\frac{D^2}{2}}f(x) =$$
$$e^{tcD}\int_{-\infty}^{\infty} f(x+y)\varphi(\frac{y}{\sqrt{t}})dy = \int_{-\infty}^{\infty} f(x+ct+y)\varphi(\frac{y}{\sqrt{t}})dy$$

where $\phi(u)$ is the standard normal density.

The hypergeometric/KWP Lie algebra. For hypergeometric processes with affine drift and quadratic volatility, the "finite computation" of the Lie group amounts to checking whether there exists a nilpotent Lie algebra containing the five components D, xD, D^2, xD^2, x^2D^2 of $\mathcal{G}_x$ and the identity I.

This is easily seen to be the case for OU processes generated by I, D, xD, D^2, even after the addition of the killing terms $x, x^2, ..., x^n$ (and in particular, the killed transition density $p(t,x,y)$ may be written down explicitly using Wei-Norman). The full KWS family is not solvable, however. Indeed, let us build the Cartan matrix of commutators, using Leibniz's rule $[a,bc] = [a,b]c + b[a,c]$ and its consequence $[a,b] = 1 \implies [a,b^n] = nb^{n-1}$, with $a = D, b = x$ and $a = -x, b = D$:

$\ddots$	x^n	D	xD	D^2	xD^2	x^2D^2
x^n	0					
D	nx^{n-1}	0	D	0		
xD	nx^n	$-D$	0	$-2D^2$	$-xD^2$	0
D^2	$n(n-1)x^{n-2} + 2nx^{n-1}D$	0	$2D^2$	0		
xD^2	$n(n-1)x^{n-1} + 2nx^nD$	$-2D^2$	xD^2	$-2D^3$	0	
x^2D^2	$n(n-1)x^n + 2nx^{n+1}D$	$-2xD^2$	0	$-2D^2 - 4xD^3$	$-2xD^2 - 2x^2D^3$	0

We note that the commutators of the first three operators belong to their vector space (and, as noticed in [86], this continues being the case if one adds the killing operators x and x^2), but this stops being the case when xD^2 is added.

Q: Note that only few Lie algebras are nilpotent/solvable. An interesting "intermediate" case is provided by the affine diffusions generated by D, xD, D^2, xD^2, for which the evolution semigroup is not nilpotent (nor solvable by quadratures), but for which the equations

$$\widehat{V}_t(t,s) = \widehat{G}\widehat{V}(t,s) + \hat{h}(s)$$

obtained by Laplace transforming in x may be solvable. It seems that the "Laplace dual operators" arising by taking Laplace transform are often "more solvable" than the original ones.

3 First passage problems

Denote by

$$\tau_L^+ = \inf\{t \geq 0;\ X_t > L\}$$
$$\tau = \tau_l = \inf\{t \geq 0;\ X_t < l\}$$

the first passage times of a stochastic process above/below given levels L, l. Computing the distribution of the latter, also called "ruin time" in the insurance literature, is one of the oldest applications of probability, introduced by Thiele, the founder of the Danish insurance company Hafnia (1872) –see www.stats.ox.ac.uk/ steffen/seminars/centertalk.pdf.

Ruin probabilities. The first objects of interest in first passage theory are the "finite-time" and "ultimate/infinite horizon" ruin probabilities $\Psi(t,x)$ and the related "survival" probabilities $\overline{\Psi}(t,x)$

$$\begin{aligned} \Psi(t,x) &= P_x[\tau \leq t], & \Psi(x) &= P_x[\tau < \infty], \\ \overline{\Psi}(t,x) &= P_x[\tau > t] = 1 - \Psi(t,x), & \overline{\Psi}(x) &= P_x[\tau = \infty] \end{aligned}$$

For the Markovian case, a first step/infinitesimal analysis shows that the ultimate ruin probabilities are **harmonic functions**, satisfying:

$$\begin{aligned}
&\mathcal{G}\Psi(u) := \frac{\sigma(x)^2}{2}\,\Psi''(u) + \varphi(x)\Psi'(u) - \lambda\Psi(u) + \\
&\lambda \int_0^u \Psi(u-z) f_C(z)dz + \lambda \bar{F}_C(u) = 0 \\
&\Psi(u) = 1, \quad u \leq 0.
\end{aligned} \tag{6}$$

The evolution equation (2) and the corresponding time-independent counterparts, the invariant measure and the harmonic functions of interest in first passage theory, have been intensively studied for **diffusions** and for **Levy processes**.

4 Piecewise deterministic processes

We study below the family of Levy driven Langevin (LL) jump processes [39]

$$\boxed{dX_t = \varphi(X_t)dt + dU_t.} \tag{7}$$

When U_t is a compound Poisson process, they reduce to the simpler family of **piecewise deterministic processes**. Particularly interesting is the case of phase-type distributed jumps

$$\bar{F}_C(x) = \int_x^\infty f_C(u)du = \boldsymbol{\beta} e^{Bx}\mathbf{1},$$

where $\mathbf{B}$ is a $n \times n$ stochastic generating matrix (nonnegative off-diagonal elements and nonpositive sow sums), and $\beta = (\beta_1, \ldots, \beta_n)$ is a row probability vector (with nonnegative elements and $\sum_{j=1}^n \beta_j = 1$), and $\mathbf{1} = (1, 1, ..., 1)$ is a column probability vector.

The Laplace transform of phase-type jumps is

$$\hat{b}(s) = \boldsymbol{\beta}(sI - \boldsymbol{B})^{-1}\boldsymbol{b},$$

with $\boldsymbol{b} = (-\boldsymbol{B})\mathbf{1}$. In this case, the convolution term from the Feynman-Kac integro-differential equation (6) may be written formally as

$$\boxed{\boldsymbol{\beta}(sI - \boldsymbol{B})^{-1}\boldsymbol{b}|_{s=D}}$$

and it may be removed by applying the differential operator $\boxed{\det(sI - \boldsymbol{B})|_{s=D}}$.

Example 3. *Consider the case of downward exponential jumps of rate μ over an exponential horizon $\mathbf{e}_q$, when the ruin probability solved the ODE*

$$\begin{aligned}
&[\varphi(x)D - \lambda - q + \lambda\frac{\mu}{\mu + D}]\Psi(x) + \lambda e^{-\mu x} = 0 \Leftrightarrow \\
&[(\mu + D)(\varphi(x)D - \lambda - q) + \lambda\mu]\Psi(x) = \\
&[\varphi(x)D^2 + (\mu\varphi(x) + \varphi'(x) - \lambda - q)D - \mu q]\Psi(x) = 0
\end{aligned}$$

Remark 2. *We will restrict to first-passage problems in domains where the drift $\varphi(x)$ doesn't change sign, which implies then the boundary conditions $\Psi(l) = 1$, provided that the drift goes towards the boundary.*

There is also a probabilistic conversion to a related continuous **embedding process**, which yields for ruin probabilities with **downward jumps** the ODE linear system

$$\begin{pmatrix} \Psi'(x) \\ \mathbf{M}'(x) \end{pmatrix} = \begin{pmatrix} \frac{\lambda+q}{\varphi(x)} & -\frac{\lambda}{\varphi(x)}\beta \\ \mathbf{b} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \Psi(x) \\ \mathbf{M}(x) \end{pmatrix}, \tag{8}$$

The variable Ψ is the killed ruin probability, q is the killing rate/Laplace transform argument, $\mathbf{b} = -B\,\mathbf{Id}$ is a column vectors, and the components $M_1, \ldots, M_n$ of the column vector $\mathbf{M}$ are killed ruin probabilities in "auxiliary stages of artificial time," introduced by changing the jumps to segments of slope ± 1.

5 Exponential jumps

Example 4. *The "embedding linear system"* (8) *in this case is:*

$$\begin{pmatrix} \Psi'(x) \\ M'(x) \end{pmatrix} = \begin{pmatrix} \frac{\lambda+q}{\varphi(x)} & -\frac{\lambda}{\varphi(x)} \\ \mu & -\mu \end{pmatrix} \begin{pmatrix} \Psi(x) \\ M(x) \end{pmatrix} \tag{9}$$

When $q = 0$, the system :

$$\begin{aligned}\Psi'(x) &= \frac{\lambda}{\varphi(x)} \left(\Psi(x) - M(x)\right) \qquad \Psi(\infty) = M(\infty) = 0 \\ M'(x) &= \mu \left(\Psi(x) - M(x)\right) \qquad\qquad M(0) = 1\end{aligned}$$

may be solved by substracting the equations, yielding:

$$\begin{aligned}\Psi(x) - M(x) &= (\Psi(0) - M(0))e^{Z(x)},\ Z(x) = -\mu x + \int_0^x \frac{\lambda}{\varphi(v)} dv \\ M(x) &= \mu(1 - \Psi(0)) \int_x^\infty e^{Z(v)} dv \qquad (10)\end{aligned}$$

and

$$\Psi(x) = M(x) + (\Psi(x) - M(x)) = (1 - \Psi(0)) \left(\mu \int_x^\infty e^{Z(v)} dv - e^{Z(x)}\right)$$

whenever $Z(\infty) = -\infty$.

Alternatively, in terms of the alternative "Riccati variable" $\eta(x) := \frac{\Psi(x)}{M(x)}$ –see below we find

$$\mu(1 - \eta(x)) = \frac{e^{Z(x)}}{\int_x^\infty e^{Z(v)} dv}{}^{\S} \qquad (11)$$

This calculation raises the question of whether Segerdahl's equation (9) (or from Example 3) is solvable by quadratures when $q > 0$. The natural framework for examining this is Lie's theory, which states that for nonautonomous systems of the form (8) to be integrable by quadratures, there must exist a "Lie system", i.e. a finitely generated (Vessiot–Guldberg) Lie algebra $\mathfrak{g}$ such that

$$A_x \equiv \begin{pmatrix} \frac{\lambda+q}{\varphi(x)} & -\frac{\lambda}{\varphi(x)}\beta \\ \mathbf{b} & \mathbf{B} \end{pmatrix} = \frac{\lambda}{\varphi(x)} \begin{pmatrix} 1+q/\lambda & -\beta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mathbf{b} & \mathbf{B} \end{pmatrix} \in \mathfrak{g}, \qquad \forall x \tag{12}$$

which is moreover solvable.

With exponential jumps and $q = 0$, integrability is determined by the family of matrices

$$A_x \equiv \begin{pmatrix} \frac{\lambda}{\varphi(x)} & -\frac{\lambda}{\varphi(x)} \\ \mu & -\mu \end{pmatrix} = \frac{\lambda}{\varphi(x)} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} = \frac{\lambda}{\varphi(x)} T_1 + \mu T_2,$$

where

$$T_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

These matrices satisfy $[T_1, T_2] = -T_1 - T_2$ and span a two dimensional solvable Lie algebra $V = \langle T_1, T_2 \rangle$, and the model is therefore integrable by quadratures, as known since Segerdahl. We show in Theorem 1 that this stops being the case when $q \neq 0$.

Theorem 1. *When $q \neq 0$, and for a non-constant drift $\varphi(x)$, the matrices A_x span the* **non-solvable** *Lie algebra $\mathfrak{gl}(2, \mathbb{R})$ of 2×2 real matrices.*

Proof. Our Lie algebra must contain $A_x = \lambda/\varphi(x)U_1 + \mu U_2$, where

$$U_1 = \begin{pmatrix} 1+q/\lambda & -1 \\ \\ 0 & 0 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix},$$

Consider the matrices

$$U_3 \equiv ([U_1, U_2] + U_2 + U_1)\lambda/q + U_2 = \begin{pmatrix} 1 & 0 \\ \\ 0 & -1 \end{pmatrix},$$

$$U_4 = [U_1, U_3] + U_2 = \begin{pmatrix} 0 & 2 \\ \\ 0 & 0 \end{pmatrix}.$$

For every Lie algebra $\mathfrak{g}$ such that $\{A_x\}_{x\in\mathbb{R}} \subset \mathfrak{g}$, the matrices U_1, U_2, U_3, U_4 must be contained in $\mathfrak{g}$, as they are made up from Lie brackets and linear combinations of elements of $\mathfrak{g}$. Moreover, as $q \neq 0$, the matrices U_1, U_2, U_3 and U_4 are linearly independent and they span $\mathfrak{gl}(2, \mathbb{R})$. It follows that $\mathfrak{gl}(2, \mathbb{R}) \subset \mathfrak{g}$. Consequently, the Lie algebra $\mathfrak{g}$ is not solvable. □

6 The Riccati approach

An alternative approach is to write the linear system (8) in the coordinate system $\{\eta = \Psi/M, M\}$, bringing it to the form

$$\begin{cases} \dfrac{\mathrm{d}\eta}{\mathrm{d}x} = -\mu\eta^2 + \left(\mu + \dfrac{\lambda+q}{\varphi(x)}\right)\eta - \dfrac{\lambda}{\varphi(x)}, \\ \dfrac{\mathrm{d}M}{\mathrm{d}x} = (\eta-1)\mu\, M. \end{cases} \tag{13}$$

The above non-linear system is made up from a homogeneous equation in the variable M and a Riccati equation in the variable η (with no dependence on the variable M), which will be called below **Segerdahl's equation**.

After the substitution $y(x) = \mu(\eta(x) - 1)$ and the homogenizing substitution $y(x) = \frac{g'(x)}{g(x)}$, the Riccati equation and it homogeneous counterpart are brought to the **canonical forms**

$$y'(x) = -y^2(x) + y(x)\left(\frac{\lambda+q}{\varphi(x)} - \mu\right) + \frac{q\mu}{\varphi(x)} \tag{14}$$
$$\Leftrightarrow g''(x) - z(x)g'(x) - u(x)g(x) = 0,$$

where

$$z(x) = \frac{\lambda+q}{\varphi(x)} - \mu, \qquad u(x) = \frac{q\mu(z(x)+\mu)}{\lambda+q}. \tag{15}$$

Note that when $q = 0$, equation (15) becomes essentially of first order $g''(x) - g'(x)z(x) = 0$, and $g'(x) = e^{Z(x)}$, with $Z(x) = \int z(x)dx$, recovering Segerdahl's result, see example 4.

Remark 3. *Note that having an explicit general solution* $\eta(x)$ *to the Riccati equation of the system* (14) *leads to an explicit general solution of the system, obtained by*

$$M(x) = L \exp\left(\int^x \eta(t)dt - \mu x\right),$$

where L is an arbitrary constant. Thus, the solution of the first-passage problem will be available analytically (up to quadratures), whenever the Riccati solution is.

6.1 The Allen-Stein family

Theorem 2. *The necessary and sufficient condition for the existence of a transformation*

$$\eta' = G(x)\eta, \quad G(x) > 0,$$

relating the Riccati equation

$$\frac{d\eta}{dx} = b_0(x) + b_1(x)\eta + b_2(x)\eta^2, \qquad b_0 b_2 \neq 0, \quad (16)$$

to an integrable one given by

$$\frac{d\eta'}{dx} = D(x)(c_0 + c_1\eta' + c_2\eta'^2), \quad c_0 c_2 \neq 0 \qquad (17)$$

where c_0, c_1, c_2 *are real numbers and* $D(x)$ *is a non-vanishing function, are*

$$D^2 c_0 c_2 = b_0 b_2, \qquad \left(b_1 + \frac{1}{2}\left(\frac{\dot{b}_2}{b_2} - \frac{\dot{b}_0}{b_0} \right) \right) \sqrt{\frac{c_0 c_2}{b_0 b_2}} = \kappa c_1, \tag{18}$$

where $\kappa = \mathrm{sg}(D) = sg(b_0/c_0)$. *The transformation is then uniquely defined by*

$$\eta' = \sqrt{\frac{b_2(x) c_0}{b_0(x) c_2}}\, \eta\,.$$

Roughly speaking, the above theorem claims that Riccati equations of the form (17) can be integrated if their coeffients b_0, b_1, and b_2 verify the condition (19).

The Riccati equation in our system (14) can be cast into the form (17), with

$$b_0(x) = -\frac{\lambda}{\varphi(x)}, \qquad b_1(x) = \left(\mu + \frac{\lambda + q}{\varphi(x)} \right), \qquad b_2(x) = -\mu. \tag{19}$$

Substituting the above functions in the Allen-Stein integrability condition (19), we get that Riccati equation (14) is integrable if the drift $\varphi(x)$ satisfies the equation

$$\boxed{\dot{\varphi}/2 + (\lambda + q) + \mu\varphi = \kappa c_1 \sqrt{-\mu\, \lambda\, c_0 c_2 \varphi}} \tag{20}$$

For example, in the particular case $c_1 = 0$, the above integrability condition reads

$$\dot{\varphi} + 2\mu\varphi + 2(\lambda + q) = 0, \tag{21}$$

whose general solution, $\varphi_0(x)$, is

$$\varphi_0(x) = \frac{\lambda + q}{\mu}\left(Ke^{-2\mu x} - 1\right),$$

with K a nonzero real constant.

An explicit solution for the classical ruin problem, i.e. the solution of the above system with initial conditions $\Psi(\infty) = M(\infty) = 0$ and $M(0) = 1$, follows.

$$\left\{\begin{aligned} \Psi(x) &= \frac{1}{(1+K_1)}\sqrt{\frac{\lambda}{q+\lambda}}(e^{2x\mu} - K)^{-1/2}(e^{x'(x)} - K_1 e^{-x'(x)}), \\ M(x) &= \frac{1}{(1+K_1)}e^{-\mu x}(K_1 e^{-x'(x)} + e^{x'(x)}). \end{aligned}\right\},$$

where

$$dx' = \sqrt{\frac{-\lambda\mu}{\varphi_0(x)}}dx \Longrightarrow$$

$$x'(x) = \frac{1}{2}\sqrt{\frac{\lambda}{q+\lambda}}\log\left(\left|\frac{1-\sqrt{|1-K|}}{\sqrt{|1-K|}+1}\frac{\sqrt{|1-e^{-2x\mu}K|}+1}{1-\sqrt{|1-e^{-2x\mu}K|}}\right|\right).$$

7 Quasi birth and death processes (QBD)

Many important stochastic models involve multidimensional random walks whose coordinates split naturally into an infinite valued coordinate ℓ called **level**, and the "rest of the information" k, called **phase**, which takes a finite number of possible values.

Partitioned according to the level, the infinitesimal generator Q of such a Markov process, is a **block tridiagonal** matrix, called level-dependent quasi-birth-and-death generator (LDQBD):

$$Q = \begin{bmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}. \tag{22}$$

QBD processes share the "skip free" structure of birth and death processes; however, the "weights" A_ℓ, B_ℓ, C_ℓ associated to each step are now matrices, inviting one to enter the noncommutative world.

The semigroup. The strongest solvability concept for continuous time Markov processes is that of the semigroup of operators e^{tQ}. Besides the straightforward case with analytically describable spectrum of Q, this may also be achievable by Lie algebra methods, if **the gen-**

erator Q may be decomposed as a sum of operators which generate a nilpotent Lie algebra. While this happens very rarely, exceptions do however exist, as shown recently by Kawanishi [**?**], who considered a special multi-server queueing model with two exponential stages of service of rates μ, μ_2, and arrival/impatience rates λ, θ, resulting in a QBD with boundary (see section 7 of [**?**]), followed by square blocks of size $(c+1)$ which depend on the level only along the diagonal:

$$A_\ell = A = \begin{pmatrix} \lambda & \dots & \dots & 0 & 0 \\ 0 & \lambda & \dots & 0 & 0 \\ 0 & \dots & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix} \qquad C_\ell = \begin{pmatrix} \ell\theta & 0 \\ \mu_2 & \ell\theta \\ 0 & \ddots \\ 0 & \dots \end{pmatrix}$$

$$B_\ell = \begin{pmatrix} * & c\mu & 0 & 0 & \dots & 0 \\ 0 & * & (c-1)\mu & 0 & \dots & 0 \\ 0 & & * & \ddots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mu \\ 0 & \cdots & \cdots & \cdots & & * \end{pmatrix}, T_\ell = \begin{pmatrix} -c\mu & c\mu & 0 & 0 \\ \mu_2 & * & (c-1)\mu & \cdot \\ 0 & 2\mu_2 & * & 2 \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & c \end{pmatrix}$$

Kawanishi noticed that generator maybe expressed in terms of "simple" matrices with a closed "multiplication table"

$$E_+ = \begin{bmatrix} 0 & c & & & \\ & 0 & c-1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}, \quad E_- = \begin{bmatrix} 0 & 0 & & & \\ 1 & 0 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & c-1 & 0 & 0 \\ & & & c & 0 \end{bmatrix},$$

$$E_-T_+ = \begin{bmatrix} 0 & 0 & & & \\ & 1 & 0 & & \\ & & 2 & 0 & \\ & & & 3 & 0 \\ & & & & 4 \end{bmatrix}, \quad [E_+, E_-] = \begin{bmatrix} c & 0 & & \\ & c-2 & 0 & \\ & & \ddots & 0 \\ & & & -c \end{bmatrix}$$

(where $[A, B] = AB - BA$ is the commutator) which generate a famous nilpotent Lie algebra. However, this model has also explicit eigenvalues (and potentially an explicit RG factorization?), rendering Lie algebra methods unnecessary here.

Even when the semigroup is not available analytically, one may hope for analytic formulas for the stationary or first passage probabilities, as indeed demonstrated by several recent results on retrial queues.

Stationary distributions. One problem of great interest for level dependent QBD processes is that of computing the stationary distribution $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots)$ partitioned by level, where $\boldsymbol{\pi}_\ell = (\pi_{\ell,0}, \pi_{\ell,1}, \ldots, \pi_{\ell,c})$. The equilibrium equations

$$\boldsymbol{\pi} Q = 0 \tag{23}$$

in partitioned form yield the second degree vector recursion:

$$\boldsymbol{\pi}_{\ell-1} A_{\ell-1} + \boldsymbol{\pi}_\ell B_\ell + \boldsymbol{\pi}_{\ell+1} C_{\ell+1} = 0; \qquad \ell = 0, 1, 2, \cdots \tag{24}$$

where π_{-1} is a vector of 0's.

The "matrix analytic" approach of Neuts is to reduce (25) to a first degree recursion

$$\boldsymbol{\pi}_\ell = \boldsymbol{\pi}_{\ell-1} R_{\ell-1}.$$

For an irreducible ergodic process, there will be a unique up to normalization matrix-product form solution

$$\boldsymbol{\pi}_\ell = \boldsymbol{\pi}_0 R_0 R_1 \cdots R_{\ell-1}, \qquad \ell = 1, 2, \ldots, \tag{25}$$

for certain matrices R_ℓ, where $\boldsymbol{\pi}_0$ is the solution (unique up to multiplication by a constant) to

$$\boldsymbol{\pi}_0 (B_0 + R_0 C_1) = 0. \tag{26}$$

An alternative for finite state space is to run the "reversed recursion"

$$\boldsymbol{\pi}_0 = \boldsymbol{\pi}_\ell \mathfrak{R}_\ell \mathfrak{R}_{\ell-1} \cdots \mathfrak{R}_1, \qquad \ell = 1, 2, \ldots, \tag{27}$$

for certain matrices $\mathfrak{R}_\ell$.

The matrices R_ℓ have been computed numerically via various methods, like cyclic and logarithmic reduction, etc. We will consider instead the possibility of obtaining analytical answers via Gaussian elimination.

8 The RG Factorizations

Gaussian elimination of a tridiagonal generator matrix Q yields the **LU/UL factorizations**:

$$\mathbf{Q} = (I - \mathbf{R_L}) \begin{bmatrix} \mathfrak{U}_0 & A_0 & & & & \\ & \mathfrak{U}_1 & A_1 & & & \\ & & \ddots & \ddots & & \\ & & & \mathfrak{U}_\ell & A_\ell & \\ & & & & \ddots & \ddots \end{bmatrix} = (I - \mathbf{R_U}) \begin{bmatrix} U_0 & & & \\ C_1 & U_1 & & \\ & \ddots & \ddots & \\ & & C_\ell & U_\ell \\ & & & \ddots \end{bmatrix}$$

see for example Faddeev and Fadeeva (Chapter 1, Section 1.13, p. 24), where

$$\mathbf{R_L} = \begin{bmatrix} 0 & & & & \\ \mathfrak{R}_1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \mathfrak{R}_N & 0 & \\ & & & \ddots & \ddots \end{bmatrix}, \qquad \mathbf{R_U} = \begin{bmatrix} 0 & R_0 & & & \\ & 0 & R_1 & & \\ & & \ddots & \ddots & \\ & & & 0 & R_N \\ & & & & \ddots \end{bmatrix}$$

where $R_j, \mathfrak{R}_j$ are precisely the recursion matrices appearing in the first order recurrences (26), (28).

The U **matrix** $U_0 = B_0 + R_0 C_1$, and more generally the matrices

$$U_\ell = B_\ell + R_\ell C_{\ell+1} := B_\ell + V_\ell$$

have a crucial probabilistic interpretation of transition rates within level ℓ of the process "censored above", i.e. observed only on levels inferior or equal to ℓ. The matrices $V_\ell := U_\ell - B_\ell$ yield thus the rates of transition "after returning from an excursion above".

A similar decomposition

$$\mathfrak{U}_\ell = B_j + \mathfrak{R}_\ell A_{\ell-1} := B_j + \mathfrak{V}_\ell$$

is available for the process "censored below", and finally

$$\boxed{\tilde{U}_\ell = B_j + V_\ell + \mathfrak{V}_\ell}$$

decomposes the transition rates of the process "censored both above and below". Note these are (semigroup) generating matrices.

The matrices $\tilde{U}_\ell$ are particular cases of "stochastic complementations" of C. Meyer (1989) [**?**], or rather stochastic completions, the latter name being inspired by the property

$$\tilde{U}_\ell \, \mathbf{1} = 0^{\S}.$$

§Note also the equations

$$(\mathfrak{U}_\ell + A_l)\,\mathbf{1} = 0, \quad (U_\ell + C_l)\,\mathbf{1} = 0,$$

which must hold, to ensure the 0 row-sums for the generators of the "censored" processes.

The stochastic completion concept is the basis of the uncoupling/aggregation approach of splitting the determination of the stationary distribution into that of the "intra-level" stationary distributions, provided by $\tilde{U}_\ell$ [§] and that of the stationary distribution of the levels, and which parallels the idea behind the matrix analytic approach.

The R-G factorizations. One very attractive computational approach for QBD's are the **block RG LDU and UDL factorizations**, which compute besides $R_\ell, \mathfrak{R}_\ell$ also the matrices $U_\ell, \mathfrak{U}_\ell$ and also the matrices $G_\ell, \mathfrak{G}_\ell$ which represent probabilistically the hitting distributions on the level below and above. The systematic use of the factorizations seems to have started only recently – see for example Quan-Lin Li and Jinhua Cao (2006) [**?**]– and these authors trace their first appearance in applied probability back to D. B. Hajek (1982) and D. Gaver, P.A. Jacobs and G. Latouche (1984) [**?**, **?**, **?**].

Theorem 3. *The generating matrice of a QBD process admits a **L**DU R-G factorization and an **U**DL R-G factorization given respectively by:*

$$\textbf{\textit{LDU:}} \quad \mathbf{Q} = (I - \mathbf{R_L})\mathbf{U}_{LDU}(I - \mathbf{G_U}) \tag{28}$$

$$\textbf{\textit{UDL:}} \quad \mathbf{Q} = (I - \mathbf{R_U})\mathbf{U}_{UDL}(I - \mathbf{G_L}) \tag{29}$$

[§] for example when $\ell = 0$, $\tilde{U}_0 = U_0$ is the generating matrix of the level 0, justifying (27) *where* $\mathbf{U}_{LDU} = diag(\mathfrak{U}_0, \mathfrak{U}_1, \cdots, \mathfrak{U}_\ell, \cdots)$, $\mathbf{U}_{UDL} = diag(U_0, U_1, \cdots, U$ *and*

$$\mathbf{G_U} = \begin{bmatrix} 0 & \mathfrak{G}_0 & & & & \\ & 0 & \mathfrak{G}_1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & \mathfrak{G}_N & \\ & & & & \ddots & \ddots \end{bmatrix} \quad \mathbf{G_L} = \begin{bmatrix} 0 & & & & & \\ G_1 & 0 & & & & \\ & \ddots & \ddots & & & \\ & & G_N & 0 & & \\ & & & \ddots & \ddots & \end{bmatrix}$$

The off-diagonal factors $\mathfrak{R}_\ell, R_\ell$ *and* $\mathfrak{G}_\ell, G_\ell$ *satisfy respectively:*

$$\mathfrak{R}_\ell(-\mathfrak{U}_{\ell-1}) = C_\ell, \ \ell = 1, ... \quad (30)$$

$$R_\ell(-U_{\ell+1}) = A_\ell, \ \ell = 0, 1, ... \quad (31)$$

and

$$(-\mathfrak{U}_\ell)\mathfrak{G}_\ell = A_\ell, \ \ell = 0, 1, ... \quad (32)$$

$$(-U_\ell)G_\ell = C_\ell, \ \ell = 1, ... \quad (33)$$

and U_ℓ *satisfy the recursions*

$$\mathfrak{U}_\ell = B_\ell - \mathfrak{R}_\ell \mathfrak{U}_{\ell-1}\mathfrak{G}_{\ell-1} = B_\ell + \mathfrak{R}_\ell A_{\ell-1} = B_\ell + C_\ell \mathfrak{G}_{\ell-1} \quad (34)$$

$$U_\ell = B_\ell - R_\ell U_{\ell+1} G_{\ell+1} = \boxed{B_\ell + R_\ell C_{\ell+1} = B_\ell + A_\ell G_{\ell+1}} \quad (35)$$

Proof:

Multiplying the matrices of the LU/UL factorizations yields respectively:

$$(I-\mathbf{R_L})\mathbf{U}_{LU}(I-\mathbf{G_U}) = \begin{bmatrix} \mathfrak{U}_0 & -\mathfrak{U}_0\mathfrak{G}_0 & & \\ -\mathfrak{R}_1\mathfrak{U}_0 & \mathfrak{U}_1+\mathfrak{R}_1\mathfrak{U}_0\mathfrak{G}_0 & -\mathfrak{U}_1\mathfrak{G}_1 & \\ \ddots & \ddots & \ddots & \\ & -\mathfrak{R}_\ell\mathfrak{U}_{\ell-1} & \mathfrak{U}_\ell+\mathfrak{R}_\ell\mathfrak{U}_{\ell-1}\mathfrak{G}_{\ell-1} & \\ & & \ddots & \end{bmatrix} \quad (36)$$

and

$$(I-\mathbf{R_U})\mathbf{U}_{UL}(I-\mathbf{G_L}) = \begin{bmatrix} U_0+R_0U_1G_1 & -R_0U_1 & & \\ -U_1G_1 & U_1+R_1U_2G_2 & -R_1U_2 & \\ \ddots & \ddots & \ddots & \\ & -U_\ell G_\ell & U_\ell+R_\ell U_{\ell+1} & \\ & & \ddots & \end{bmatrix} \quad (37)$$

The equality of the secondary and main diagonals yields then immediately the result.

■

Remark 4. *The R and G matrices satisfy second order recurrences, which are respectively:*

$$\mathfrak{R}_{\ell+1}\mathfrak{R}_\ell A_{\ell-1}+\mathfrak{R}_{\ell+1}B_\ell+C_{\ell+1} = 0, \Leftrightarrow \mathfrak{R}_{\ell+1} = -C_{\ell+1}[\mathfrak{R}_\ell A_{\ell-1}+B_\ell]^{-1}$$

$$R_\ell R_{\ell+1}C_{\ell+2}+R_\ell B_{\ell+1}+A_\ell = 0, \Leftrightarrow R_\ell = -A_\ell[R_{\ell+1}C_{\ell+2}+B_{\ell+1}]^{-1} \quad \ell =$$

$$C_\ell \mathfrak{G}_{\ell-1} \mathfrak{G}_\ell + B_\ell \mathfrak{G}_\ell + A_\ell = 0, \Leftrightarrow \quad \ell = 1, ...$$

$$A_\ell G_{\ell+1} G_\ell + B_\ell G_\ell + C_\ell = 0, \Leftrightarrow \quad \ell = 1, ...$$

Like any second order bilinear recurrence, these may be solved in principle by iterating backwards the "matrix-continued fraction type" recursions above.

Truncation choices. While the LU factorization is straightforward to implement recursively from its initial condition $\mathfrak{U}_0 = B_0$, the UL factorization for infinite state processes requires in practice truncation to a finite number of levels L, and running the recursion downwards from L. Some ad-hoc initialization of the matrix U_L, which may also be interpreted as an "adjustment" rendering 0 the sum of the rows of truncated process censored above L (getting thus an ergodic approximation).

Two possible adjustments are to modify the diagonal of U_L (which represents the last diagonal block in the generator of the process censored above the level L), so that the last rows add up to 0, by taking it as

$$U_L = B_L + A_L,$$

or as

$$U_L = B_L + Diag(A_L)$$

(the same adjustments for U_L will also work for $\mathfrak{U}_L$).

We will call the latter, corresponding to just canceling the arrivals to level $L+1$ **simple reflection truncation**. Note that for BD processes, this is the only possible "ergodic truncation", and that for QBD process with diagonal arrivals A_ℓ, the two procedures coincide.

Any truncation will reduce the problem to solving a **finite linear system**, which may be solved symbolically! However, in level dependent problems, the results will depend on L and on the truncation method adopted.

Consider a **"G-truncation" procedure generalizing the classic approximations of Fallin and Neuts-Rao**

$$U_L = B_L + A_L \tilde{G}_{L+1,K,k} \Leftrightarrow \tilde{U}_L = \mathfrak{U}_L + A_L \tilde{G}_{L+1,K,k} \quad (38)$$

where $\tilde{G}_{L,K,k}$ denote the first passage probabilities to level L for the process for which K levels above L follow the Neuts-Rao approximation (a fixing of the retrial rates) and the next k levels follow the Falin approximation (changing the orbit into an instant access queue).

Note that since this requires only solving a linear first passage system, the generalized Fallin-Neuts-Rao truncation may also be implemented **symbolically**!

Some results with the "simple G-truncation" $\tilde{G}_{L,1,0}$ are reported below.

Remark 5. *Even though the LU factorization produces the* $\mathfrak{U}_\ell, ...$ *matrices without ad-hoc intervention, to compute the stationary distribution we must obtain the value of* π_L*, and this will again require adjusting* $\mathfrak{U}_L$ *so that it becomes a generating matrix; thus, a stochastic completion of* $\mathfrak{U}_\ell$ *will finally be required.*

Example 5. Birth-death processes.

The LU factorization yields $G_i = 1, i = 0, ...$*'s and* $\mathfrak{R}_i = \frac{\mu_i}{\lambda_{i-1}}, i = 1, ...$ *(the reciprocals of the "classic" R's).*

The UL *factorization with "ergodic" truncation yields* $R_i = \frac{\lambda_i}{\mu_{i+1}}, i = 0, ..., G_i = 1.$

The "classic" quadratic equations for R, G *of the M/M/1 queue are:*

$$\mu R^2 - (\lambda + \mu)R + \lambda = 0, \lambda G^2 - (\lambda + \mu)G + \mu = 0,$$

with roots $\{R \to 1, R \to \frac{\lambda}{\mu}\}$, $\{G \to 1, G \to \frac{\mu}{\lambda}\}$.

In the ergodic case we have $G = 1$*, and it follows from* $-U_\ell = GC$ *that* $-U = \mu$*, and that* $R = \rho := \frac{\lambda}{\mu}$ *(the probabilistic interpretation is easily verified, since the number of visits one level above is geometric with parameter* $p = \frac{\rho}{1+\rho}$ *and its expectation* $\frac{p}{1-p}$ *simplifies to* ρ*).*

Remark 6. *The* G_ℓ *of the UL decomposition is the matrix of probabilities of ever moving one level below.*

More precisely

$$G_\ell(i,j) = P_{(\ell,i)}[\tau_{\ell-1} < \infty, X(\tau_{\ell-1}) = (\ell-1, j)]$$

(G_ℓ must be therefore a stochastic/substochastic matrix, in the ergodic/nonergodic case).

Indeed, let $\tilde{B}_\ell$ denote the matrix of transition rates within the same level, with the diagonal set to 0, let $T = A_\ell + \tilde{B}_\ell + C_\ell$, and let $Diag(T)$ denote a diagonal matrix containing the sums of the rows of T. Conditioning after dt yields

$$G_\ell = A_\ell dt G_{\ell+1} G_\ell + \tilde{B}_\ell dt G_\ell + (I - Diag(T) dt) G_\ell \Leftrightarrow A_\ell G_{\ell+1} G_\ell + (\tilde{B}_\ell -$$

which yields the corresponding recursion, since $B_\ell = \tilde{B}_\ell - Diag(T)$.

Similarly, $\mathfrak{G}_\ell$ of the UL decomposition is the **stochastic matrix of probabilities of ever moving one level up**.

Remark 7. *The matrices $\{R_\ell\}_{\ell \geq 0}$ of the UL decomposition have a more complicated probabilistic interpretation – see for example [**?**]. The (i,j)th entry $(R_\ell)_{i,j}$ of R_ℓ is the expected sojourn time in the state $(\ell+1, j)$, given the process started in state (ℓ, i), before the first revisit of level ℓ, and divided by $-B_\ell(i,i)$, which is the expected sojourn time in the state (ℓ, i), before leaving it.*

*More precisely, the following formula holds [**?**]:*

$$(R_\ell)_{i,j} = q_\ell(i,j)\frac{B_\ell(i,i)}{B_\ell(j,j)} \tag{39}$$

In discrete time, the fraction equals 1 *and* $R_\ell)_{i,j} = q_\ell(i,j)$ *is the expected number of visits to* $(\ell+1,j)$ *before returning to level* ℓ*, given the process started in state* (ℓ,i)*.*

Remark 8. *In the homogeneous (level independent) case), it is clear from the probabilistic interpretations and verifiable through algebra that the factorization matrices will not depend on the level, and will satisfy quadratic Riccati equations. In the* UL *case, these are respectively:*

$$\mathfrak{R}^2 A + \mathfrak{R}B + C = 0, \qquad R^2C + RB + A = 0$$

$$C\mathfrak{G}^2 + B\mathfrak{G} + A = 0, \qquad AG^2 + BG + C = 0$$

In the asymptotically convergent case, we can obtain the limits $R = \lim_n R_n, G = \lim_n G_n$ *(if they exist), via the same second order equations. One reasonable strategy in the* UL *case is to start by computing the* G *limit and then, iterating backwards, the* G_ℓ *matrices.*

Remark 9. *When* $\mathfrak{U}_\ell, U_\ell$ *are invertible, the off-diagonal factors* R_ℓ *and* G_ℓ *may be obtained by:*

$$\mathfrak{R}_\ell = C_\ell(-\mathfrak{U}_{\ell-1}^{-1}), \ell = 1, ... \quad (40)$$

$$R_\ell = A_\ell(-U_{\ell+1}^{-1}),\ \ell = 0, 1, ... \quad (41)$$

and

$$\mathfrak{G}_\ell = -(\mathfrak{U}_\ell^{-1})A_\ell, \ell = 0, 1, ... \quad (42)$$

$$G_\ell = -(U_\ell^{-1})C_\ell,\ \ell = 1, ... \quad (43)$$

and U_ℓ *satisfy the recursions*

$$\mathfrak{U}_\ell = B_\ell - C_\ell \mathfrak{U}_{\ell-1}^{-1} A_{\ell-1} \quad (44)$$

$$U_\ell = B_\ell - A_\ell U_{\ell+1}^{-1} C_{\ell+1} \quad (45)$$

Contents. Below, we illustrate via some examples the fact that the RG factorization yield in a systematic way many of the analytical results already obtained in the literature, including calculations of Laplace transforms of transient distributions.

We rederive via the RG factorizations the results of Liu and Zhao [**?**] for the $M/M/c/c$ retrial queue, with $c = 1, 2$, by adopting a "simple truncation"– see below. For $c = 1, 2$ the truncation effect (the dependence of L) disappears after one iteration. When ≥ 3 however, the simple truncation Mathematica results depend on L in a complicated way (fractions whose degree explodes with the truncation level).

In level independent cases however, the dependence on L and on the truncation disappears typically after a few iterations, yielding thus easily classic results for priority models, Kawanishi's model, the M/M/1 queue with feedback, etc.

9 M/M/c/c + R retrial models

Retrial queues. An important example of QBD's are multiserver retrial queues, for which interesting **analytic results** were obtained by Y.C. Kim (1995), B.D. Choi & al(1998) and A. Gómez-Corral and M.F. Ramalhoto (1999) [**?**, **?**, **?**], and more recently, by Liu and Zhao ($c = 2$), and Phung-Duc & al ($c = 3, 4$) [**?**, **?**] [§]

The retrial model with geometric loss $\alpha \leq 1$, acceptance $p \leq 1$ and feedback $\beta \geq 0$ is a QBD with a simple linear dependence on the level

$$A_\ell = A, \quad C_\ell = \ell C, \quad B_\ell = B - \tilde{A} - \ell \tilde{C}$$

where $\tilde{A}, \tilde{C}$ denote the sum of the diagonals of A, C, with

[§]Note that it was widely believed that an explicit expression for the joint probability distribution when $c > 3$ does not exist (see, for example pp.25 of [**?**] and also pp.288 of [**?**]), and hints that this belief may be wrong appeared only recently [**?**, **?**].

QBD structure defined by the square blocks of size $(c+1)$:

$$C = \begin{bmatrix} 0 & \nu & 0 & \dots & 0 \\ 0 & 0 & \nu & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \nu \\ 0 & \dots & \dots & \dots & \bar{\alpha} \end{bmatrix} \qquad A_\ell = A = \begin{bmatrix} \lambda\bar{p} & \dots & \dots & 0 \\ \mu\beta & \lambda\bar{p} & \dots & 0 \\ 0 & 2\mu\beta & \dots & 0 \\ \vdots & \ddots & \ddots & \lambda\bar{p} \\ 0 & \dots & \dots & c\mu\beta \end{bmatrix}$$

and

$$B = \begin{bmatrix} -\lambda p & \lambda p & 0 & 0 & \dots & 0 \\ \mu\bar{\beta} & -(\lambda p + \mu\bar{\beta}) & \lambda p & 0 & \dots & 0 \\ 0 & 2\mu\bar{\beta} & -(\lambda p + 2\mu\bar{\beta}) & \lambda p & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda p \\ 0 & \cdots & \cdots & \cdots & c\mu\bar{\beta} & -c\mu\bar{\beta} \end{bmatrix}$$

where $\bar{\alpha} = 1 - \alpha$, ...

The three matrices of interest may be written as:

$$A = \lambda\alpha M + \lambda\bar{p}(I - M) + \mu\beta E_-, \quad C = \nu T_+ + \bar{\alpha} M,$$
$$B = \mu\bar{\beta}(E_- - E_- T_+) + \lambda p(T_+ - (I - M)).$$

The Lie algebra they generate is **not nilpotent?**

Note that B is a generating matrix and that the phase generating matrix $T_\ell := A_\ell + B_\ell + C_\ell = (\ell\nu + \lambda p)(T_+ - (I - M)) + \mu(E_- - E_- T_+)$ has indeed sum of rows 0, as it should.

We examine next the classic case $\alpha = p = 1, \beta = 0$.

9.1 The generating function approach to classic retrial queues

We will consider only stable systems, with $\lambda < c\mu$, which ensures the existence of stationary probabilities. Introducing the generating function:

$$p_k(z) = \sum_{\ell=0}^{\infty} \pi_{\ell,k} z^{\ell}, \tag{46}$$

multiplying the equilibrium equations by z^{ℓ}, and summing up gives rise to the first order differential system:

$$\mathbf{p}'(z)V(z) = \mathbf{p}(z)U(z) \tag{47}$$

where $\mathbf{p}(z) = (p_0(z), \cdots, p_c(z))$, $\mathbf{p}'(z) = (p_0'(z), \cdots, p_c'(z))$, and $V(z)$, $U(z)$ are square matrices of order $(c+1)$:

$$V(z) = z\tilde{C} - C = \nu \begin{bmatrix} z & -1 & 0 & \dots & 0 \\ 0 & z & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & z & -1 \\ 0 & \dots & \dots & \dots & \frac{\bar{\alpha}(z-1)}{\nu} \end{bmatrix},$$

$$U(z) = B+(z-1)A = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ \mu & -(\lambda+\mu) & \lambda & 0 & \dots \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -(\lambda+(c- \\ 0 & \dots & \dots & \dots & c\mu \end{bmatrix}$$

Remark 10. *This matrix has an explicit LU decomposition $M(z) = lu$ where*

$$l = I - \frac{\mu}{\lambda}EM = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\mu}{\lambda} & 1 & 0 & 0 \\ 0 & -\frac{2\mu}{\lambda} & 1 & 0 \\ 0 & 0 & -\frac{3\mu}{\lambda} & 1 \end{pmatrix},$$

$$u = \lambda(T_+ - I + zM) = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & (z-1)\lambda \end{pmatrix}$$

For $c = 3$ for example, the differential system is

$$\begin{cases} \mu p(1)(z) = (\lambda + \nu z D)p(0)(z), \\ 2\mu p(2)(z) = (\lambda + \mu + \nu z D)p(1)(z) - (\lambda + \nu D)p(0)(z), \\ 3\mu p(3)(z) = (\lambda + 2\mu + \nu z D)p(2)(z) - (\lambda + \nu D)p(1)(z), \\ (\lambda + 3\mu - z\lambda)p(3)(z) - (\lambda + \nu D)p(2)(z) = 0 \end{cases} \tag{48}$$

Or, after multiplying by the inverse of u and using

$$V_1(z) = V(z)u^{-1} = \begin{pmatrix} -\frac{\nu z}{\lambda} & \frac{\nu-\nu z}{\lambda} & \frac{\nu-\nu z}{\lambda} & \frac{\nu}{\lambda} \\ 0 & -\frac{\nu z}{\lambda} & \frac{\nu-\nu z}{\lambda} & \frac{\nu}{\lambda} \\ 0 & 0 & -\frac{\nu z}{\lambda} & \frac{\nu}{\lambda} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(49) becomes

$$\begin{cases} -p(0)(z) + \frac{\mu p(1)(z)}{\lambda} - \frac{\nu z p(0)'(z)}{\lambda} = 0, \\ -p(1)(z) + \frac{2\mu p(2)(z)}{\lambda} + \frac{(\nu-\nu z)p(0)'(z)}{\lambda} - \frac{\nu z p(1)'(z)}{\lambda} = 0, \\ -p(2)(z) + \frac{3\mu p(3)(z)}{\lambda} + \frac{(\nu-\nu z)p(0)'(z)}{\lambda} + \frac{(\nu-\nu z)p(1)'(z)}{\lambda} - \frac{\nu z p(2)'(z)}{\lambda} = 0, \\ -p(3)(z) + \frac{\nu p(0)'(z)}{\lambda} + \frac{\nu p(1)'(z)}{\lambda} + \frac{\nu p(2)'(z)}{\lambda} = 0 \end{cases} \quad (4$$

Note from the last equation that $p(c)(z)$ is just the sum of the derivatives of the other unknowns – see also the related (57).

Eliminating now $p(1)(z)$ from the first equation of (49), $p(2)(z)$ from the second, etc substituting in the last equation and putting $\tilde{\lambda} = \frac{\lambda}{\nu}, \tilde{\mu} = \frac{\mu}{\nu}$ yields the scalar equation

$$\tilde{\lambda}^4 p(0)(z) + (fz+g)\, p(0)'(z) + \left(cz^2 + dz + e\right) p(0)''(z) \\ +z^2(z\tilde{\lambda} - 3\tilde{\mu})p(0)^{(3)}(z) = 0 \quad (50)$$

where

$f = \tilde{\lambda}\left(3\tilde{\lambda}^2 + 3\tilde{\lambda}(\tilde{\mu}+1) + 2\tilde{\mu}^2 + 1 + 3\tilde{\mu}\right), g = -\tilde{\mu}\left(6\tilde{\lambda}^2 + 8(\tilde{\mu}+1)\tilde{\lambda}\right.$

$$c = 3\tilde{\lambda}(\tilde{\lambda} + \tilde{\mu} + 1), d = -9\tilde{\mu}(\tilde{\lambda} + \tilde{\mu} + 1), e = 3\tilde{\mu}^2$$

(note the singularities coefficient is in general $z^{c-1}(z\tilde{\lambda} - c\tilde{\mu})$).

Remark 11. *Linear systems with polynomial coefficients may be always automatically "uncoupled" to triangular form, for example by Gaussian elimination, by Abramov-Zima elimination, or by Zurcher's algorithm, which brings the system to Frobenius block companion matrix form. Finally, this reduces the problem to solving scalar equations with polynomial coefficients, which may be factored sometimes for example by Maple (OreTools).*

The system (48) has been solved analytically only for $c = 1, 2$ –see for example [**?**, **?**, **?**], but not for higher values.

For $c = 1$, the scalar equation is

$$p(0)(z)\lambda^2 + (z\lambda - \mu)\nu p(0)'(z) = 0 \tag{51}$$

with solution proportional to

$$(1 - \rho z)^{-\tilde{\lambda}}$$

where $\rho = \frac{\lambda}{c\mu} = \frac{\lambda}{\mu} < 1$. It follows from the last equation in (50) that $p(1)(z)$ is proportional to

$$\rho(1 - \rho z)^{-\tilde{\lambda}-1}.$$

Using $p(0)(z)+p(1)(z)|_{z=1} = 1$ yields the proportionality constant $(1-\rho)^{1+\tilde{\lambda}}$.

Let us review now Hanshke's [**?**] solution for $c = 2$, when the scalar equation is

$$\lambda^3\pi(0)(z) + \nu(z\lambda(2\lambda + \ \mu + \nu) - \mu(3\lambda + 2(\mu+\nu)))\pi(0)'(z) + z(z\lambda -$$

Putting $\rho z = x$ yields the Gauss hypergeometric equation

$$x(x-1)y''(x)+[x(2\tilde{\lambda}+\tilde{\mu}+1)-(\frac{3\tilde{\lambda}}{2}+\tilde{\mu}+1)]y'(x)+\tilde{\lambda}^2y(x) = 0$$

whose only analytic solution in the unit disk is the Gauss hypergeometric function. This determines all unknowns up to a proportionality constant, obtained using $p(0)(z)+p(1)(z)+p(2)(z)|_{z=1} = 1$.

Remark 12. *The fact that the retrial model is a QBD with constant matrices A, B and a simple linear dependence on the level $C_\ell = \ell C$ implies that both the stationary distributions and their generating functions satisfy holonomic systems for any c. Obtaining however initial conditions is not immediate.*

One possibility is of obtaining the recurrence and values for the stationary distribution of phases

$$\pi_k = \sum_\ell \pi_{\ell,k} = p(k)(1)?$$

Another would be using the fact that $\pi(0)(z)$ is analytic at $z = 0$, which provides $c - 1$ additional constraints for the initial conditions?

9.2 The G, U and R matrices

A first hint on an extra special structure here was provided by the fact that for arbitrary c, in the level independent case (with total constant retrial rate), the R matrix intervening in the matrix-geometric solution for the stationary distribution has all rows 0, except the last one [**?**][1]. Furthermore, as an immediate consequence of (42) and of the fact only the last row in A_k is non-zero, this remains true in the level independent case, i.e.

$$R_\ell = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ r_{\ell,0} & r_{\ell,1} & \cdots & r_{\ell,c} \end{bmatrix} \tag{53}$$

The special structure R_ℓ implies the proportionality of $\boldsymbol{\pi}_\ell$ to the last row of $R_{\ell-1}$.

Theorem 4. *For the M/M/c retrial queue given in (23), the stationary probability vector* $\boldsymbol{\pi}$ *can be ex-*

[1]The constant retrial rate simplification has given rise to several "generalized truncation" approximations (Section 2.5 of [**?**], [**?**] and [**?**])

pressed as

$$\boldsymbol{\pi}_\ell = \pi_{0,c} r_{0,c} r_{1,c} \cdots r_{\ell-2,c}(r_{\ell-1,0}, r_{\ell-1,1}, \ldots, r_{\ell-1,c}), \qquad \ell = 1, 2, \ldots, \tag{54}$$

where $\boldsymbol{\pi}_0$ is uniquely determined by the equation (27) and the normalizing condition. Alternatively,

$$\pi_{\ell,j} = \pi_{\ell-1,c} r_{\ell-1,j}$$

We recall now some general equations derived in [**?**] by censoring, and which may also be obtained by multiplying the matrix recurrences by conveniently chosen vectors, and by the generating function approach.

Lemma 1. *For the $M/M/c$ retrial queue, we have, putting $\nu_\ell = (\ell + 1)\nu$.*

1.

$$(\lambda + \ell\nu)\pi_{\ell,0} \;=\; \mu\pi_{\ell,1}, \Leftrightarrow (\lambda + \ell\nu) r_{\ell-1,0} = \mu r_{\ell-1,1}, \qquad \ell = 1, 1, 2$$

2.

$$r_{\ell,0} + r_{\ell,1} + \cdots + r_{\ell,c-1} = \frac{\lambda}{\nu_\ell}, \qquad \ell = 0, 1, 2, \ldots. \tag{56}$$

3.

$$\begin{cases} \lambda(r_{\ell,1} - r_{\ell,c} + 1) - 2\mu r_{\ell,2} - \nu_\ell(\sum_{k=2}^{c-1} r_{\ell,k}) + \nu_{\ell+1} r_{\ell,c}(\sum_{k=1}^{c-1} r_{\ell+1,} \\ \lambda(r_{\ell,2} - r_{\ell,c} + 1) - 3\mu r_{\ell,3} - \nu_\ell(\sum_{k=3}^{c-1} r_{\ell,k}) + \nu_{\ell+1} r_{\ell,c}(\sum_{k=2}^{c-1} r_{\ell+1,} \\ \ldots \\ \lambda(r_{\ell,c-2} - r_{\ell,c} + 1) - ((c-1)\mu + \nu_\ell) r_{\ell,c-1} + \nu_{\ell+1} r_{\ell,c}(\sum_{k=c-2}^{c-1} r_{\ell} \\ \lambda(r_{\ell,c-1} - r_{\ell,c} + 1) - c\mu r_{\ell,c} + \nu_{\ell+1} r_{\ell,c} r_{\ell+1,c-1} \end{cases}$$

Remark 13. *When $c = 2$, the first two equations determine already $r_{\ell,0}$, $r_{\ell,1}$, $\ell = 0, 1, \ldots$ as*

$$r_{\ell,0} = \frac{\lambda}{\nu_\ell} \frac{\mu}{\lambda + \mu + \nu_\ell} = \frac{\lambda}{\nu_\ell} G_\ell(2,1), \quad r_{\ell,1} = \frac{\lambda}{\nu_\ell} \frac{\lambda + \nu_\ell}{\lambda + \mu + \nu_\ell} = \frac{\lambda}{\nu_\ell} G_\ell(2,2)$$

The single remaining equation in 3. yields then

$$r_{\ell,c} = \frac{\lambda(1 + r_{\ell,c-1})}{\lambda + c\mu - \nu_{\ell+1} r_{\ell+1,c-1}} = \frac{\lambda}{\mu\,\nu_\ell} \frac{\lambda + \mu + \nu_{\ell+1}}{\lambda + \mu + \nu_\ell} \frac{\mu\nu_\ell + (\lambda + \nu_\ell)^2}{3\lambda + 2\mu + 2\nu_{\ell+1}}.$$

Proof. 1) Multiplying the recurrence

$$R_\ell(B_{\ell+1} + R_{\ell+1} C_{\ell+2}) + A_\ell = 0 \tag{58}$$

for the R equation by the unique eigenvector $e_1 := (1, 0, 0, \ldots)$ of the eigenvalue 0 of the C_ℓ matrix changes the recurrence into:

$$R_\ell B_{\ell+1} e_1 = 0 \Leftrightarrow (\lambda + \nu_\ell) r_{\ell,0} = \mu r_{\ell,1} \tag{59}$$

2) Multiply now the recurrence (59) by $\mathbf{1} := (1, 1, 1, \ldots)$. Putting

$$s_\ell := r_{\ell,0} + r_{\ell,1} + \cdots + r_{\ell,c-1}$$

and using $B_{\ell+1}\mathbf{1} = (-\nu_j, -\nu_j, -\nu_j, ..., -\nu_j, -\lambda)$ yields in the last row

$$\nu_\ell s_\ell r_{\ell,c} - \lambda = (s_{\ell+1}\nu_{\ell+1} - \lambda) = ... = 0, \qquad (60)$$

by iterating towards ∞ and using the ergodicity (this result may be found already in [**?**]).

3) follows by multiplying the recurrence (59) by $\boldsymbol{e}_i := (0, 0, 1, ..., 1)$ which sums the last i rows, $i = 1, c-1$.

9.3 Symbolic factorization results for retrial queues

Figure 1: States and transitions of the $M/M/3/3$ retrial queue

Remark 14. *The simplest family here are the* $\mathfrak{G}_\ell$ *matrices, which are* 0*, except for a last column of ones*

$\mathfrak{G}(i,j) = \delta_{c+1}(j)$ *(since when the level increases, all servers must be working). This implies that* $\mathfrak{V}$ *is a matrix of* 0, *except the first* c *elements of the last column, which equal* $\ell\nu$ *(the row sums of* C_ℓ*). Finally, we obtain the rational expression (though rather complicated as a function of* c*)*

$$\mathfrak{R}_\ell = -C_\ell(\mathfrak{U}_{\ell-1})^{-1}.$$

For the UDL factorization, the R_ℓ *matrices have a special structure with* **only the last row nonzero** *(54), as a direct outcome of the fact that if at least one server is idle, i.e. the system starts in a state* (ℓ, i) *with* $0 \leq i \leq c-1$, *the number of customers in the orbit can not increase without making all servers busy first, and so the return to level* ℓ *must happen with* 0 *time spent above!*

For the UDL factorization we find that the "completion matrices"

$$V_\ell = R_\ell C_{\ell+1} = A_\ell G_{\ell+1}$$

have the first $c-1$ rows and first column 0. The last non zero row is:

$$V_\ell(c+1,*) = \begin{pmatrix} 0 & \boldsymbol{v}_\ell \end{pmatrix} \quad \text{where } \nu_\ell = (\ell+1)\nu, \quad \boldsymbol{v}_\ell = \nu_\ell \boldsymbol{r}_j,$$

and $\boldsymbol{r}_\ell = (r_{\ell,0}, ..., r_{\ell,c-1})$ denotes the first c elements of the last row of R_ℓ.

Note that the matrices $G_\ell(\ell = 1, \cdots, N)$ must have the first column 0, since first passage transitions to the states $(\ell, 0)$ is impossible, and that by the last equality in (46), it holds that $\frac{\nu_\ell}{\lambda}\boldsymbol{r}_\ell$ equals the non zero part of the last row of $G_{\ell+1}$. Thus, the matrices R_ℓ follow immediately from $G_{\ell+1}$, except for the corner $R_\ell c, c$.

For $c = 1$, the factorization finds:

$$G_\ell = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad \forall \ell = 1, \cdots {}^{\S}. \tag{61}$$

The matrices $R_\ell(\ell = 0, \cdots,)$ are:

$$R_\ell = \begin{pmatrix} 0 & 0 \\ r_{\ell,0} & r_{\ell,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{\lambda}{\nu_\ell} & \frac{\lambda(\lambda+\nu_\ell)}{\mu\nu_\ell} \end{pmatrix} \tag{62}$$

where $\nu_\ell = (\ell + 1)\nu$.

After determining $\pi_{0,1}$ by censoring at 0,, and the normalization condition, we find:

Corollary 1. *For the standard $M/M/1$ retrial queue, the stationary distribution is given by*

$$\begin{aligned} \boldsymbol{\pi}_\ell &= \pi_{0,1} r_{1,1} r_{2,1} \cdots r_{\ell-1,1}(r_{\ell,0}, r_{\ell,1}) \\ &= \pi_{0,1} \frac{(\lambda/\mu)^\ell}{\ell!} \left(\prod_{k=1}^{\ell} (\lambda/\nu + k) \right) \left(\frac{\mu}{\lambda + n\nu},\ 1 \right), \quad n \geq 1, \end{aligned} \tag{63}$$

§This formula is obvious probabilistically, since $G_\ell(i, 0) = 0, i = 0, 1, \forall \ell$ forces $G_\ell(i, 1) = 1$ (algebraically, this may be checked to equal 1 from the linear system at level L, with reflection boundary condition, and can also be easily verified to propagate to the other levels).

and

$$\boldsymbol{\pi}_0 = (\pi_{0,0}, \pi_{0,1}) = \pi_{0,1}\left(\frac{\mu}{\lambda}, 1\right), \qquad (64)$$

where

$$\pi_{0,1} = \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right)^{\frac{\lambda}{\nu}+1}.$$

It is easy to check that the above result is consistent with that given on page 3 in [**?**].

For **the** $M/M/2/2$ **retrial queue** with $p = \alpha = 1, \beta = 0$, the UDL factorization finds the **exact rational expression**

$$G_\ell = \frac{1}{\lambda+\mu+\ell\nu}\begin{pmatrix} 0 & \mu+\ell\nu & \lambda \\ 0 & \mu & \lambda+\ell\nu \\ 0 & \mu & \lambda+\ell\nu \end{pmatrix} \qquad \forall \ell = 1, \cdots, \qquad (65)$$

either via a simple reflection or a simple G-truncation [§] .

Based on (66), it seems natural to guess with c servers a general perturbation expansion

$$a^{(c)}(\ell) G_\ell = \sum_{i=0}^{c-1} b_i \ell^i G(i) \qquad (66)$$

where $a^{(c)}(\ell)$ is a polynomial of degree $c-1$ and $G(i)$ are constant matrices.

[§]Note that the last two rows must be equal, since first hitting the lower level at $j=1$ or $j=2$ forces the descent starting point (by retrial) to be $i=0$ and $i=1$, respectively, and so $G(i,j)$ equal the proportions of long run time spent in $i=0$ and $i=1$, which are $\frac{\mu}{\lambda+\mu+\ell\nu}$ and $\frac{\lambda+\ell\nu}{\lambda+\mu+\ell\nu}$.

Indeed, when $c = 3$, the G recurrence yields...

The final result for $c = 2$ is:

Lemma 2. *For the standard $M/M/2$ retrial queue, the stationary distribution is given by*

$$\begin{aligned}\boldsymbol{\pi}_\ell &= \pi_{0,2} r_{0,2} r_{1,2} \cdots r_{\ell-2,2}(r_{\ell-1,0}, r_{\ell-1,1}, r_{\ell-1,2}) \\ &= \pi_{0,2} \frac{1}{(\ell-1)!} \left(\frac{\lambda}{\nu\ mu}\right)^{\ell-1} \left(\prod_{k=0}^{\ell-2} \frac{\nu_k \mu + (\lambda+\nu_k)^2}{3\lambda + 2\mu + 2\nu_{k+1}}\right) \frac{\lambda+\mu+\nu_\ell}{\lambda+\mu+\nu}(r\end{aligned}$$

and

$$\boldsymbol{\pi}_0 = (\pi_{0,0}, \pi_{0,1}, \pi_{0,2}) = \pi_{0,2} \left(\frac{\mu^2}{\lambda^2} \frac{3\lambda+2\mu+2\nu}{\lambda+\mu+\nu}, \frac{\mu}{\lambda} \frac{3\lambda+2\mu+2\nu}{\lambda+\mu+\nu}, 1\right), \tag{68}$$

where

$$\pi_{0,2} = ...$$

(is determined by the normalization condition).

For the **M/M/3/3 retrial queue**, the Mathematica results with simple truncation are very complicated, depending on the truncation level. However, the first "simple G-approximation" is pretty simple:

$$\tilde{G}_{\ell,1,0} = \begin{pmatrix} 0 & \frac{2\mu^2+3j\nu\mu+j\nu(\lambda+j\nu)}{(\lambda+\mu+j\nu)^2+\mu(\mu+j\nu)} & \frac{\lambda(2\mu+j\nu)}{(\lambda+\mu+j\nu)^2+\mu(\mu+j\nu)} & \frac{\lambda^2}{(\lambda+\mu+j\nu)^2+\mu(\mu+j\nu)} \\ 0 & \frac{\mu(2\mu+j\nu)}{(\lambda+\mu+j\nu)^2+\mu(\mu+j\nu)} & \frac{(\lambda+j\nu)(2\mu+j\nu)}{(\lambda+\mu+j\nu)^2+\mu(\mu+j\nu)} & \frac{\lambda(\lambda+j\nu)}{(\lambda+\mu+j\nu)^2+\mu(\mu+j\nu)} \\ 0 & \frac{2\mu^2}{(\lambda+\mu+j\nu)^2+\mu(\mu+j\nu)} & \frac{2\mu(\lambda+j\nu)}{(\lambda+\mu+j\nu)^2+\mu(\mu+j\nu)} & \frac{\lambda^2+2j\nu\lambda+j\nu(\mu+j\nu)}{(\lambda+\mu+j\nu)^2+\mu(\mu+j\nu)} \\ 0 & \frac{2\mu^2}{(\lambda+\mu+j\nu)^2+\mu(\mu+j\nu)} & \frac{2\mu(\lambda+j\nu)}{(\lambda+\mu+j\nu)^2+\mu(\mu+j\nu)} & \frac{(\lambda+j\nu)^2+j\nu\mu}{(\lambda+\mu+j\nu)^2+\mu(\mu+j\nu)} \end{pmatrix}$$

suggesting that the first c elements of the last row of R_j might be given by:

$$\frac{\nu_j}{\lambda\gamma}\big(2\mu^2, 2\mu(\lambda+\nu_j), (\lambda+\nu_j)^2+\nu_j\mu)\big)$$

where $\gamma = (\lambda+\mu+\nu_j)^2 + \mu(\mu+\nu_j)$.

In conclusion, we find that while the truncated problems for fixed L reduce to linear systems, easy symbolically, computing the limit when $L \to \infty$ of this procedure may be quite hard. By "luck", the classic results valid when $c = 1, 2$ and $\lambda_0 = 0$ are "essentially" independent of L, for several truncations.

10 Complex exponential transforms, cf. Jacobson and Jensen

The classical analytical approach via Laplace transforms suffers from certain difficulties: for example, for first-passage downwards of spectrally negative processes, the presence of jumps up invalidated the approach, but not the answer. One way out was to use sometimes Fourier transform instead of Laplace transform.

Recently, a new light on these difficulties was shed by Jacobsen and Jensen [55], in the case of generalized Ornstein-Uhlebeck processes, with a fixed lower boundary l and $L = \infty$, by employing a classic method of solving differential equations with polynomial coefficients. This method, originating with Poincaré, which consists in looking for solutions of the form

$$\ominus(x) = \int_\Gamma e^{xz}\widehat{\ominus}(z)dz = \int_\Gamma e^{xz}z^{-1}\ominus^*(z)dz \tag{69}$$

where the "kernel" $\widehat{\ominus}(z)$ and the integration contour Γ are yet to be determined.

Plugging this into the equation (79), and applying the Lagrange identity to the diffusion part operator, one finds that two conditions must be satisfied:

1. The kernel $\widehat{\ominus}(z)$ must satisfy a "dual Laplace" operator (83)

$$\widehat{G}\widehat{\ominus}(z) = 0$$

2. To remove boundary effects, the integration contour Γ must be closed, or connect zeros of the bilinear concomitant of the diffusion operator (see for example Ince, ch VIII, XVIII [53], and [55], Prop.2).

Definition 1. *Primitive killed eigenfunctions are solutions of the "Sturm-Liouville" equation*

$$(\mathcal{G} - q(x))\ominus_\Gamma(x) = 0,$$

which may also be represented as complex (Laplace-type) exponential transforms along **connected integration contours**

$$\ominus_\Gamma(x) = \int_\Gamma \widehat{\ominus}(z)e^{xz}dz$$

where $\widehat{\ominus}(z)$ is a homogeneous solution of the "dual Laplace" operator and where the contour is chosen so that the boundary contributions cancel.

For example, $\widehat{\ominus}(z)$ might be the usual Laplace transform, in which case Γ would be a Bromwich contour.

Example 6. *The general affine case $a_0 > 0, a_1 > 0$ may be reduced to GCIR by choosing $-\frac{a_0}{a_1}$ as origin. The kernel takes different forms in the remaining two cases:*

$$\ominus^*(x) = x^{-\tilde{q}} \begin{cases} e^{r^{-1}\int_0^x \frac{\kappa(u)}{u}du} & \text{for } a_1 = 0 \\ B(x)^{\frac{c}{a_1}+\tilde{q}-1}e^{-\int_0^x \frac{\lambda\widehat{\overline{F}}(z)+\lambda_{(+)}\widehat{\overline{F}}_{(+)}(z)}{B(z)}dz} & \text{for } a_1 > 0, a_0 = 0 \end{cases} \tag{70}$$

where $\lambda, \lambda_+\widehat{\widehat{F}}, \widehat{\widehat{F}}_+$ represent the intensities and Laplace transforms of the negative and positive jumps, and where $B(x) = r + a_1 x$ (the GOU case with $a_1 = q = 0$ appeared already in Hadjiev [51]). Note in both cases the appearance of the term

$$J(x) = e^{\int_0^x \frac{\lambda\widehat{\widehat{F}}(z)+\lambda_{(+)}\widehat{\widehat{F}}_{(+)}(z)}{B(z)}dz},$$

which depends on the jump part only, with the exception of the linear term $B(x)$.

In the GOU case with phase-type jumps up and down, (71) *becomes:*

$$\ominus^*(x) = x^{-\tilde{q}} \prod_{k=1}^{K} (z + \mu_k)^{-\lambda\alpha_k/r} \prod_{k=1}^{K_+} (z - \nu_k)^{-\lambda_{(+)}\alpha_{(+),k}/r}. \quad (71)$$

Remark 15. *It turns out that a "miracle" takes place, with* **phase-type jumps***: the cardinality of the maximal number of linearly independent primitive Sturm-Liouville functions equals that of non-equivalent integration contours, and the number of possible ways of crossing the boundary downwards. Thus a linear system may be set up, whose number of unknowns is equal with the number of equations, which yields the Gerber-Shiu function* (**??**)*!*

Note furthermore that when $r > 0$, the origin and the rates μ_k of the phases crossing downwards give rise to singularities in (72)*, which suggests for poles small circles surrounding them as integration contours, or Bromwich contours avoiding branch cuts otherwise, while when $r < 0$ these produce zeroes of the kernel $\ominus^*$, which allows real integration contours connecting them to be used.*

This distinction is very important, since the GOU process is stationary or transient, depending on whether $r < 0/r > 0$, and the [55] paper are the first to translate this probabilistic difference into an analytic one.

The final conclusion [55], Thm 4, is that the GS function is given by

$$E_x[e^{-q\tau+\xi(X_\tau - l)}] = \sum_j c_j \int_{\Gamma_j} e^{xz}\widehat{\ominus}(z)dz + e^{\xi(x-l)} I_{x<l},$$

where $\boldsymbol{c} = (c_1, c_2, ...)$ satisfies the system $M\boldsymbol{c} = \mathbf{1}$, with

$$M_{i,j} = (\mu_i + \xi) \int_{\Gamma_j} \frac{\widehat{\ominus}(z)}{\mu_i + z} e^{-lz} dz.$$

11 Affine processes

The moment generating function at a fixed time. Let us recall first the essential tool of Levy semigroups, the formula of the Laplace transform with respect to the initial position

$$E_x e^{\xi(X_t - x)} = e^{t\kappa(\xi)},$$

which defines the "symbol"/cumulant generating function $\kappa(\xi)$.

As well-known, [38], the extension to affine processes requires the solution of a generalized Riccati equation.

Lemma 3. *Let X_t be an affine process, i.e. a Markovian process whose operator has a Levy-Khinchine decomposition with coefficients affine in the initial state:*

$$a(x) = a_0 + a_1 x, \quad \varphi(x) = c + rx, \quad \nu(x, z) = \nu_0(z) + x\nu_1(z).$$

Let $\kappa(x)$ denote the symbol of the Levy process $X_t^{(0)}$ obtained by setting to 0 the first order coefficients a_1, r, ν_1.

Let $q(x) = q_1 x$ denote a linear discount function. Then, the logarithm of the joint transform $E_x e^{-\int_0^t q(X_s)ds + \xi X_t}$ is also affine in the initial state x, i.e.

$$E_x e^{-\int_0^t q(X_s)ds + \xi X_t} = e^{x\phi(t,\xi) + \Phi(t,\xi)}.$$

Putting

$$B(x) = a_1 x + r, \tag{72}$$

the functions ϕ, Φ can be obtained from the SDE by solving a generalized Riccati equation

$$\begin{aligned} &\frac{\partial}{\partial t}\phi(t,\xi) = a_1\phi^2(t,\xi) + r\phi(t,\xi) - q_1 := \phi(t,\xi)B(\phi(t,\xi)) - q_1, \\ &\phi(0,\xi) = \xi \end{aligned} \tag{73}$$

and by an integration:

$$\Phi(t,\xi) = \int_0^t \kappa(\phi(s,\xi))ds \tag{74}$$

Example 7. *For the GOU process, the Riccati equation is:*

$$\frac{\partial}{\partial t}\phi(t,\xi) = r\phi(t,\xi) - q_1, \quad \phi(0,\xi) = \xi \tag{75}$$

with solution

$$\phi(t,\xi) = \xi e^{rt} - q_1 \frac{e^{rt}-1}{r},$$

and

$$\mathbb{E}_x \left(e^{-q_1 \int_0^t X_s ds + \xi X_t - x\phi(t,\xi)} \right) = e^{\int_0^t (\kappa(\phi(s,\xi)))ds}$$

With $q_1 = 0$ we get:

$$\mathbb{E}_x \left(e^{\xi X_t - x e^{rt}} \right) = e^{\int_0^t \kappa(\xi e^{rs})ds} = e^{\frac{1}{r}\int_\xi^{\xi e^{rt}} \frac{\kappa(u)}{u} du}$$

which appears already in Hadjiev [51].

Remark 16. *While the limit $\lim_{t\to\infty} X_t$ might exist or not for the GOU process, depending on $r < 0/r > 0$, the quantity*

$$\Phi_\infty(\xi) = lim_{t\to\infty} \int_0^t \kappa(\phi(s,\xi))ds = \begin{cases} r^{-1} \int_\xi^\infty \frac{\kappa(u)}{u} du & \text{if } r > 0 \\ -r^{-1} \int_0^\xi \frac{\kappa(u)}{u} du & \text{if } r < 0 \end{cases}$$

exists in both cases and will play an important role below. For example, for any $q > 0$, let

$$M_t = \int_0^\infty e^{\xi X_t + \Phi_\infty(\xi) - qt} \xi^{-\frac{q}{r}-1} d\xi.$$

We may check, putting $z = \xi_t = \xi e^{rt}$, that

$$E_x[M_t] = \int_0^\infty e^{x\xi_t + \Phi_\infty(\xi_t) - qt} \xi^{-\frac{q}{r}-1} d\xi = \int_0^\infty e^{xz + \Phi_\infty(z)} z^{-\frac{q}{r}-1} dz = M_0$$

and furthermore that M_t is a martingale.

Q 1: Similar, but more complicated formulas, are available in the CIR case.

Q2: Provide extensions to KWPS processes, and examine the "solvability" of these jump-diffusions from "Lie's perspective".

12 Double Laplace transforms for hypergeometric processes

Let us consider now spectrally negative KWP diffusions with jumps, and introduce the operator:

$$(GV)(x) := [\sum_{k=1}^{2}\sum_{i=0}^{k} a_i^{(k)} x^i \, D^k]V(x) + \lambda \int_0^x (V(x-u) - V(x)) f(u) du. \quad (76)$$

Consider the expected payoff at ruin

$$V(t,u) = \mathbb{E}_{\{X_0=u\}} \left(p(X_t) \; 1_{\{\tau \geq t\}} + w\,(X_\tau) \; 1_{\{\tau < t\}} \right) \quad (77)$$

where $u = X_0 \geq 0$ are the initial reserves, and w, p represent respectively:

- The penalty at ruin $w(X_\tau)$ with deficit X_τ, $\quad w : \mathbb{R}^- \to \mathbb{R}^-$
- The reward or pay-off on survival after t years: $p(X_t)$, $P : \mathbb{R}^+ \to \mathbb{R}^+$.

and analyze the dividend + penalty Gerber-Shiu function $V_q(x) = W_q(x) + \mathcal{D}(x)$ on an interval $[0, B]$, which satisfies the system:

$$\begin{aligned}
&G \; V_q(x) - (\lambda + q) V_q(x) + h(x) = 0, \quad \text{for } x \geq 0 \qquad (78)\\
&V_q(0) = w(0_-) \qquad \text{if } \sigma > 0\\
&a(0, D) V_q(0) = -w_\nu(0) - qp(0) + w(0_-)(\lambda + q)\\
&V_q'(B) = 1
\end{aligned}$$

where

$$w_\nu(x) = \int_x^\infty w(x-u) \mathfrak{V}(du), \quad h(x) = w_\nu(x) + qp(x). \quad (79)$$

denote the expected jump payoff starting from x and a combination of the two payoffs.

Ignoring at first the last equation, we will solve first the system independent of B up to a proportionality constant, to be determined finally from the last equation.

As in classical ruin theory, the first step will be to obtain an equation for the Laplace transform in the initial reserves, resulting in a "Laplace dual" operator –see (83).

Lemma 4. *a) When $B = \infty$, the double Laplace transform of the expected penalty at ruin:*

$$\widehat{V}_q(s) = \int_0^\infty e^{-sx}\, V_q(x)dx = \int_{x=0}^\infty e^{-sx} \int_0^\infty qe^{-qt}V(t,x)dt\, dx \tag{80}$$

satisfies for $s > 0$ the linear ODE:

$$\begin{aligned}\widehat{G}\widehat{V}_q(s) &:= a_2 \left(s^2\, \widehat{V}_q(s)\right)'' - a_1 \left(s^2\, \widehat{V}_q(s)\right)' - r\left(s\, \widehat{V}_q(s)\right)' + (\kappa(s) - q)\, \widehat{V}_q(s) \\ &+ \widehat{h}(s) = V_q(0)\,(c + a_0 s - a_1) + a_0\, V_q'(0) = \tilde{V}_q(0) + a_0 w(0_-)s \end{aligned} \tag{81}$$

where $\boxed{\tilde{V}_q(0) = V_q(0)\,(c - a_1) + a_0\, V_q'(0)}$ *and where the "Laplace dual" operator satisfied by the Laplace transform, given by*

$$\widehat{G} = \sum_{k=0}^{2} \sum_{i=0}^{k} a_i^{(k)}\, (-D_s)^i [s^k\ \ ] + \lambda \hat{f}(s) \tag{82}$$

may be read out directly from the formula for G

$$G = \sum_{k=0}^{2} \sum_{i=0}^{k} a_i^{(k)} x^i\, (D_x)^k + \lambda f(x)* \tag{83}$$

see [53].

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