Which first passage problems are solvable?

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1 M/M/c/c + R retrial models

Retrial queues. An important example of QBD's are multiserver retrial queues, for which interesting **analytic results** were obtained by Y.C. Kim (1995), B.D. Choi & al(1998) and A. Gómez-Corral and M.F. Ramalhoto (1999) [**?**, **?**, **?**], and more recently, by Liu and Zhao ($c = 2$), and Phung-Duc & al ($c = 3, 4$) [**?**, **?**] [§]

The retrial model with geometric loss $\alpha \leq 1$, acceptance $p \leq 1$ and feedback $\beta \geq 0$ is a QBD with a simple linear dependence on the level

$$A_\ell = A, \quad C_\ell = \ell C, \quad B_\ell = B - \tilde{A} - \ell \tilde{C}$$

where $\tilde{A}, \tilde{C}$ denote the sum of the diagonals of A, C, with QBD structure defined by the square blocks of size $(c + 1)$:

$$C = \begin{bmatrix} 0 & \nu & 0 & \dots & 0 \\ 0 & 0 & \nu & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \nu \\ 0 & \dots & \dots & \dots & \bar{\alpha} \end{bmatrix} \qquad A_\ell = A = \begin{bmatrix} \lambda\bar{p} & \dots & \dots & 0 & 0 \\ \mu\beta & \lambda\bar{p} & \dots & 0 & 0 \\ 0 & 2\mu\beta & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \lambda\bar{p} & \vdots \\ 0 & \dots & \dots & c\mu\beta & \lambda\alpha \end{bmatrix}$$

and

$$B = \begin{bmatrix} -\lambda p & \lambda p & 0 & 0 & \dots & 0 \\ \mu\bar{\beta} & -(\lambda p + \mu\bar{\beta}) & \lambda p & 0 & \dots & 0 \\ 0 & 2\mu\bar{\beta} & -(\lambda p + 2\mu\bar{\beta}) & \lambda p & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \lambda p \\ 0 & \cdots & \cdots & \cdots & c\mu\bar{\beta} & -c\mu\bar{\beta} \end{bmatrix}$$

[§] Note that it was widely believed that an explicit expression for the joint probability distribution when $c > 3$ does not exist (see, for example pp.25 of [**?**] and also pp.288 of [**?**]), and hints that this belief may be wrong appeared only recently [**?**, **?**].

where $\bar{\alpha} = 1 - \alpha, ...$

The three matrices of interest may be written as:

$$A = \lambda\alpha M + \lambda\bar{p}(I - M) + \mu\beta E_-, \quad C = \nu T_+ + \bar{\alpha}M,$$
$$B = \mu\bar{\beta}(E_- - E_-T_+) + \lambda p(T_+ - (I - M)).$$

The Lie algebra they generate is **nilpotent, of dimension ?**

Note that B is a generating matrix and that the phase generating matrix $T_\ell := A_\ell + B_\ell + C_\ell = (\ell\nu + \lambda p)(T_+ - (I - M)) + \mu(E_- - E_-T_+)$ has indeed sum of rows 0, as it should.

The generating function approach. Introducing the generating function:

$$p_k(z) = \sum_{\ell=0}^{\infty} \pi_{\ell,k} z^\ell, \tag{1}$$

multiplying the equilibrium equations by z^ℓ, and summing up gives rise to the first order differential system:

$$\mathbf{p}'(z)V(z) = \mathbf{p}(z)U(z) \tag{2}$$

where $\mathbf{p}(z) = (p_0(z), \cdots, p_c(z))$, $\mathbf{p}'(z) = (p'_0(z), \cdots, p'_c(z))$, and $V(z)$, $U(z)$ are square matrices of order $(c+1)$:

$$V(z) = z\tilde{C} - C = \nu \begin{bmatrix} z & -1 & 0 & \dots & 0 \\ 0 & z & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & z & -1 \\ 0 & \dots & \dots & \dots & \frac{\bar{\alpha}(z-1)}{\nu} \end{bmatrix},$$

$$U(z) = B + (z-1)A = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & \dots & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -(\lambda+(c-1)\mu) & \lambda \\ 0 & \cdots & \cdots & \cdots & c\mu & \lambda\alpha(z-1) - c\mu \end{bmatrix}$$

Remark 1. *This matrix has an explicit LU decomposition* $U(z) = \mathfrak{l}\mathfrak{u}$. *When* $c = 3$,

$$\mathfrak{l} = I - \frac{\mu}{\lambda} EM = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\mu}{\lambda} & 1 & 0 & 0 \\ 0 & -\frac{2\mu}{\lambda} & 1 & 0 \\ 0 & 0 & -\frac{3\mu}{\lambda} & 1 \end{pmatrix},$$

$$\mathfrak{u} = \lambda(T_+ - I + zM) = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & \lambda\alpha(z-1) \end{pmatrix}$$

After multiplying by the inverse of $\mathfrak{u}$, *the system becomes:*

$$\mathbf{p}'(z) V_1(z) = \mathbf{p}(z)\mathfrak{l} \tag{3}$$

where

$$V_1(z) = V(z)\mathfrak{u}^{-1} = \begin{pmatrix} -\frac{\nu z}{\lambda} & \frac{\nu - \nu z}{\lambda} & \frac{\nu - \nu z}{\lambda} & \frac{\nu}{\lambda\alpha} \\ 0 & -\frac{\nu z}{\lambda} & \frac{\nu - \nu z}{\lambda} & \frac{\nu}{\lambda\alpha} \\ 0 & 0 & -\frac{\nu z}{\lambda} & \frac{\nu}{\lambda} \\ 0 & 0 & 0 & \frac{\bar{\alpha}}{\lambda\alpha} \end{pmatrix}$$

Note from the last equation that $p(c)(z)$ *is just the sum of the derivatives of the other unknowns – see also the related* (**??**).

We examine next the classic case $\alpha = p = 1, \beta = 0$.

1.1 The classic retrial queues

We will consider only stable systems, with $\lambda < c\mu$, which ensures the existence of stationary probabilities.

Since $V(z)$ is not invertible now, it is convenient to eliminate the last component $p_c(z)$, for example from the last equation of (3) $p_c(z) = \frac{(\nu D)\sum_{i=0}^{c-1} p_i(z)}{\lambda}$. Letting $\boldsymbol{\pi}(z) = (p_0(z), ...p_{c-1}(z))$ denote the first c unknowns, the system (3) becomes

$$\boldsymbol{\pi}'(z)\big(- \nu z \boldsymbol{u} + \nu \boldsymbol{u}_1 + \frac{c\mu\nu}{\lambda}\boldsymbol{l}\big) = \boldsymbol{\pi}(z)\big(\lambda \boldsymbol{I} - E_-\big), \tag{4}$$

where $\boldsymbol{u} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad \boldsymbol{u}_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\quad \boldsymbol{l} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

For $c = 3$ for example, the differential system (2) is

$$\begin{cases} \mu p(1)(z) = (\lambda + \nu z D)p(0)(z), \\ 2\mu p(2)(z) = (\lambda + \mu + \nu z D)p(1)(z) - (\lambda + \nu D)p(0)(z), \\ 3\mu p(3)(z) = (\lambda + 2\mu + \nu z D)p(2)(z) - (\lambda + \nu D)p(1)(z), \\ (\lambda + 3\mu - z\lambda)p(3)(z) - (\lambda + \nu D)p(2)(z) = 0 \end{cases} \tag{5}$$

the system (3) is

$$\begin{cases} -p_0(z) + \frac{\mu p_1(z)}{\lambda} - \frac{\nu z p_0'(z)}{\lambda} = 0, \\ -p_1(z) + \frac{2\mu p_2(z)}{\lambda} + \frac{(\nu - \nu z)p_0'(z)}{\lambda} - \frac{\nu z p_1'(z)}{\lambda} = 0, \\ -p_2(z) + \frac{3\mu p_3(z)}{\lambda} + \frac{(\nu - \nu z)p_0'(z)}{\lambda} + \frac{(\nu - \nu z)p_1'(z)}{\lambda} - \frac{\nu z p_2'(z)}{\lambda} = 0, \\ -p_3(z) + \frac{\nu p_0'(z)}{\lambda} + \frac{\nu p_1'(z)}{\lambda} + \frac{\nu p_2'(z)}{\lambda} = 0 \end{cases} \tag{6}$$

and the reduced system (4), after solving for the first derivatives, is:

$$\boldsymbol{\pi}'(z) = \boldsymbol{\pi}'(z) \begin{pmatrix} -\frac{\lambda}{z\nu} & \frac{(z-1)\lambda}{z^2\nu} & \frac{(z-1)\lambda}{z^3\nu} \\ \frac{\mu}{z\nu} & -\frac{\lambda}{z\nu} - \frac{(z-1)\mu}{z^2\nu} & \frac{(z-1)\lambda}{z^2\nu} - \frac{(z-1)\mu}{z^3\nu} \\ 0 & \frac{2\mu}{z\nu} & -\frac{\lambda}{z\nu} - \frac{2(z-1)\mu}{z^2\nu} \end{pmatrix} \tag{7}$$

There is a singularity at $z = 0$, and the method of solution is affected by the type of singularity. To classify it as regular or irregular, we must switch to a scalar form, because of the absence of a Fuchs-type criterion for systems.
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Eliminating $p_1(z)$ from the first equation of (5), $p_2(z)$ from the second, etc substituting in the last equation and putting $\tilde{\lambda} = \frac{\lambda}{\nu}, \tilde{\mu} = \frac{\mu}{\nu}$ yields the scalar equation

$$\begin{aligned} &\tilde{\lambda}^4 p(0)(z) + (fz + g)\, p(0)'(z) + \left(cz^2 + dz + e\right) p(0)''(z) \\ &+ z^2(z\tilde{\lambda} - 3\tilde{\mu})p(0)^{(3)}(z) = 0 \end{aligned} \tag{8}$$

where

$$f = \tilde{\lambda}\left(3\tilde{\lambda}^2 + 3\tilde{\lambda}(\tilde{\mu}+1) + 2\tilde{\mu}^2 + 1 + 3\tilde{\mu}\right), g = -\tilde{\mu}\left(6\tilde{\lambda}^2 + 8(\tilde{\mu}+1)\tilde{\lambda} + 3\left(2\tilde{\mu}^2 + 3\tilde{\mu} + 1\right)\right),$$

$$c = 3\tilde{\lambda}(\tilde{\lambda} + \tilde{\mu} + 1), d = -9\tilde{\mu}(\tilde{\lambda} + \tilde{\mu} + 1), e = 3\tilde{\mu}^2$$

(note the singularities coefficient is in general $z^{c-1}(z\tilde{\lambda} - c\tilde{\mu})$).

Remark 2. *Linear systems with polynomial coefficients may be always automatically "uncoupled" to triangular form, for example by Gaussian elimination, by Abramov-Zima elimination, or by Zurcher's algorithm, which brings the system to Frobenius block companion matrix form. Finally, this reduces the problem to solving scalar equations with polynomial coefficients, which may be factored sometimes for example by Maple (OreTools).*

The system (**??**) has been solved analytically only for $c = 1, 2$ –see for example [**?**, **?**, **?**], but not for higher values.

For $c = 1$, the scalar equation is

$$p(0)(z)\lambda^2 + (z\lambda - \mu)\nu p(0)'(z) = 0 \tag{9}$$

with solution proportional to

$$(1 - \rho z)^{-\tilde{\lambda}}$$

where $\rho = \frac{\lambda}{c\mu} = \frac{\lambda}{\mu} < 1$. It follows from the last equation in (6) that $p(1)(z)$ is proportional to

$$\rho(1 - \rho z)^{-\tilde{\lambda}-1}.$$

Using $p(0)(z)+p(1)(z)|_{z=1} = 1$ yields the proportionality constant $(1-\rho)^{1+\tilde{\lambda}}$.

Let us review now Hanshke's [**?**] solution for $c = 2$, when the scalar equation is

$$\lambda^3\pi(0)(z) + \nu(z\lambda(2\lambda + \mu + \nu) - \mu(3\lambda + 2(\mu + \nu)))\pi(0)'(z) + z(z\lambda - 2\mu)\nu^2\pi(0)''(z) = 0 \tag{10}$$

Putting $\rho z = x$ yields the Gauss hypergeometric equation

$$x(x - 1)y''(x) + [x(2\tilde{\lambda} + \tilde{\mu} + 1) - (\frac{3\tilde{\lambda}}{2} + \tilde{\mu} + 1)]y'(x) + \tilde{\lambda}^2 y(x) = 0$$

whose only analytic solution in the unit disk is the Gauss hypergeometric function. This determines all unknowns up to a proportionality constant, obtained using $p(0)(z) + p(1)(z) + p(2)(z)|_{z=1} = 1$.

Remark 3. *The fact that the retrial model is a QBD with constant matrices A, B and a simple linear dependence on the level $C_\ell = \ell C$ implies that both the stationary distributions and their generating functions satisfy holonomic systems for any c. Obtaining however initial conditions is not immediate.*

One possibility is of obtaining the recurrence and values for the stationary distribution of phases

$$\pi_k = \sum_{\ell} \pi_{\ell,k} = p(k)(1)?$$

Another would be using the fact that $\pi(0)(z)$ is analytic at $z = 0$, which provides $c - 1$ additional constraints for the initial conditions?

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