Ingham-Beurling inequalities, number theory and control of PDE's

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Outline

- Background on exact observability and its cost
- Two examples
- Statement of the main results
- Ingham-Beurling type inequalities
- How does analytic number theory help

Background on exact observability and its cost

Let X and Y be Hilbert spaces, $A : \mathcal{D}(A) \to X$ et $C \in \mathcal{L}(\mathcal{D}(A), Y)$.

$$(1) \qquad \dot{w}(t) = Aw(t), \quad y(t) = Cw(t).$$

Assume that A generates a C^0 semigroup, denoted $\mathbb{T}$, in X.

Definition 1. $C \in \mathcal{L}(\mathcal{D}(A), Y)$ is an *admissible observation operator for* $\mathbb{T}$ if there exist $\tau > 0$, $k_\tau > 0$ such that

$$k_\tau^2 \int_0^\tau \|C\mathbb{T}_t z_0\|_U^2 \mathrm{d}t \leq \|z_0\|_X^2 \qquad \forall\, z_0 \in \mathcal{D}(A).$$

Definition 2. Let $\tau > 0$ and let $C \in \mathcal{L}(\mathcal{D}(A), Y)$ be an admissible observation operator for $\mathbb{T}$. The pair (A, C) is *exactly observable in time* τ if there exists $K_\tau > 0$ such that

$$K_\tau^2 \int_0^\tau \|C\mathbb{T}_t z_0\|_U^2 \mathrm{d}t \geq \|z_0\|_X^2 \qquad \forall\, z_0 \in \mathcal{D}(A). \tag{1}$$

Definition 3. The smallest constant $K_\tau > 0$ satisfying (1) is said the observation cost in time τ.

Remark 1. C is admissible iff the operators $(\Psi_\tau)_{\tau>0}$ defined by

$$(\Psi_\tau z_0)(t) = \begin{cases} C\,\mathbb{T}_t\, z_0 & \text{for } \; t \in [0,\tau]\,, \\ 0 & \text{for } \; t > \tau\,, \end{cases}$$

can be extended to operators in $\mathcal{L}(X, L^2(0,\infty;Y))$.
(A, C) is exactly observable in time τ iff

$$K_\tau \|\Psi_\tau z_0\|_{\mathcal{L}(\mathcal{D}(A),L^2(0,\infty;Y))} \geq \|z_0\|_X \qquad \forall\, x \in \mathcal{D}(A)\,.$$

Proposition 1. If (A, C) is exactly observable in time τ then there exists the operators $E_\tau \in \mathcal{L}(L^2(0,\infty;Y), X)$ such that

$$I_X = E_\tau \Psi_\tau, \quad K_\tau = \|E_\tau\|\,.$$

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Two examples

A one-dimensional Schrödinger equation

$$\begin{cases} \dot{w} + i\frac{\partial}{\partial x}\left(a\frac{\partial w}{\partial x}\right) = 0 & (x \in (0,\pi),\ t \geq 0), \\ w = 0 & (x \in \{0,1\},\ t \geq 0), \\ w = w_0 & (x \in (0,1),\ t = 0). \end{cases}$$

The exact boundary observability problem:

$$k_\tau^2 \int_0^\tau \left| \frac{\mathrm{d}w}{\mathrm{d}x}(0,t) \right|^2 \mathrm{d}t \geq \int_0^\pi \left| \frac{\mathrm{d}w_0}{\mathrm{d}x}(0) \right|^2 \mathrm{d}x$$

Equivalent form : If (λ_n) is a regular sequence and $\tau > 0$ (large enough) then there exists k_τ with

$$k_\tau^2 \int_0^\tau \left| \sum_{n\geq 0} a_n e^{i\lambda_n t} \right|^2 \mathrm{d}t \geq \sum_{n\geq 0} |a_n|^2. \qquad ((a_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C}))$$

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The Schrödinger equation in a square

Let $\Omega = [0, \pi]^2$, $\Gamma \subset \partial\Omega$ and $\tau > 0$.
The required observability inequality is

$$K_{\tau,\Gamma}^2 \int_0^\tau \int_\Gamma \left| \frac{\partial w}{\partial \nu} \right|^2 \mathrm{d}\Gamma \geq \int_\Omega |\nabla w(0)|^2 \mathrm{d}x\mathrm{d}t,$$

for every solution w.

$$\begin{cases} \dot{w} + i\Delta w = 0 & (x \in \Omega,\ t \geq 0), \\ w = 0 & (x \in \partial\Omega,\ t \geq 0). \end{cases}$$

Remark.
The classical result of Lebeau (1992) needs a Γ containing two sides.

An equivalent inequality

Let τ, $a > 0$. The observability inequality holds for every τ and Γ iff for every τ and L

$$\int_0^{\tau}\int_0^{L}\Big|\sum_{m,n\in\mathbb{Z}} a_{mn}\mathrm{e}^{2\pi i\left(nx+\{m^2+n^2\}t\right)}\Big|^2 \mathrm{d}x\mathrm{d}t \geq \delta \sum_{m,n\in\mathbb{Z}} |a_{mn}|^2$$

for every $(a_{mn}) \in \ell^2(\mathbb{Z}\times\mathbb{Z},\mathbb{C})$.

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Previous cost estimates (1D)

Result of Miller (2006):

$$\liminf_{\tau \to 0} \tau \ln k_\tau \geq \frac{\mu}{4},$$

$$\limsup_{\tau \to 0} \tau \ln k_\tau \leq 4 \left(\frac{36}{37} \right)^2 \mu,$$

where $\mu := \left(\int_0^\pi \sqrt{a(x)} \mathrm{d}x \right)^2$.

Open question: Improve the upper bound and possibly show that

$$\lim_{\tau \to 0} \tau \ln k_\tau, = \frac{\mu}{4}.$$

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Previous results in 2D

For boundary observation:
Two sides are required.

For observation distributed inside Ω:
Any observation region is OK (Jaffard, 1992).
No longer true if Ω is a disk.

Statement of the main results

First main result (I)

Theorem. (Tenenbaum and Tucsnak, to appear in JDE)
In the 1D case, the observation cost k_τ satisfies

$$\limsup_{\tau \to 0} \tau \ln k_\tau \leq \frac{3\mu}{4},$$

where $\mu := \left(\int_0^\pi \sqrt{a(x)} \mathrm{d}x \right)^2$.

Remark. The above improve Miller's estimate

$$\limsup_{\tau \to 0} \tau \ln k_\tau \leq 4 \left(\frac{36}{37} \right)^2 \mu$$

but we are still far from $\frac{1}{4}$.

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Second main result

Theorem. (Tenenbaum and Tucsnak to appear in Trans. of the AMS)
Let Ω be a rectangle in $\mathbb{R}^2$ and let $\tau > 0$. Then the system is exactly observable in any time $\tau > 0$ iff Γ contains non empty open vertical and horizontal parts.
Moreover, if Ω is square-like then there exist $K_1,\ K_2,\ K_3$ s.t.

$$K_{\tau,\Gamma} \leq \exp\left\{K_1 \frac{(\ln|I_1|)^2}{|I_1|} + K_2 \frac{(\ln|I_2|)^2}{|I_2|} + \mathrm{e}^{K_3/\tau}\right\}$$

Remark. The result improves Ramdani, Takahashi, Tenenbaum et Tucsnak (JFA, 2006) where we tackled the case of te square without getting the arbirarily small observation time.

Inequalities of Ingham-Beurling-Kahane type and cost estimates

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Ingham's inequality

Theorem. (Ingham, 1936) Let $\gamma,\ M > 0$ and let $(\lambda_n) \in \ell^2(\mathbb{Z}, \mathbb{R})$ s.t.

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0 \qquad (n \in \mathbb{Z}).$$

Then, for every interval I with $|I| > 2\pi/\gamma$ and for every sequence $(a_n) \in \ell^2(\mathbb{Z}, \mathbb{C})$, we have

$$\left(|I| - \frac{2\pi}{\gamma}\right) \sum_{n\in\mathbb{Z}} |a_n|^2 \leq \int_I \left|\sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t}\right|^2 dt \leq \left(|I| + \frac{2\pi}{\gamma}\right) \sum_{n\in\mathbb{N}} |a_n|^2.$$

Proof

We set $f(t) = \sum a_n \mathrm{e}^{i\lambda_n t}$ with $\gamma = 1$, $I = [-\pi - \epsilon, \pi + \epsilon]$.

Let $k(t) = \frac{1+\cos t}{\pi^2 - t^2}[(\pi + \epsilon)^2 - t^2]$.

Then $k \in L^1(\mathbb{R}) \cup L^\infty(\mathbb{R})$ is an even function and $K = \widehat{k}$, with

$$k(t) \leq 0 \;\text{ si }\; t \notin [-\pi - \epsilon, \pi + \epsilon],$$

$$K(u) = 0 \;\text{ si }\; |u| \geq 1, \quad K(0) > 0.$$

Then

$$K(0) \sum |a_n|^2 = \sum \sum a_n \overline{a_m} K(\lambda_m - \lambda_n) = \int_{-\infty}^{+\infty} k(t)|f(t)|^2 dt$$

$$\leq \int_{-\pi-\epsilon}^{\pi+\epsilon} k(t)|f(t)|^2 dt \leq C \int_{-\pi-\epsilon}^{\pi+\epsilon} |f(t)|^2 dt,$$

Theorem. (Beurling, 1950, Tenenbaum et Tucsnak (2006)).
Let $(\lambda_n) \subset \mathbb{R}$ be a sequence with

$$\lambda_{n+1} - \lambda_n \ \geq \ \gamma', \qquad (n \in \mathbb{Z}),$$

Assume that there exists $\gamma \geq \gamma'$ and $M \in \mathbb{N}^*$ such that

$$\lambda_{n+M} - \lambda_n \ \geq \ \gamma M \qquad (n \in \mathbb{N}).$$

Then, for every interval I with $l(I) > \dfrac{2\pi}{\gamma}$, there exists $\delta > \delta(\gamma, \gamma', M, I)$ such that

$$\int_I \left| \sum_{n\in\mathbb{Z}} a_n \, e^{i\lambda_n x} \right|^2 \mathrm{d}x \ \geq \ \delta \sum_{n\in\mathbb{Z}} |a_n|^2,$$

for every $(a_n) \subset l^2(\mathbb{C})$.

A particular class of non-harmonic Fourier series

Proposition (Tenenbaum et Tucsnak, 2007) Let $(\lambda_n) \subset \mathbb{R}$ be a regular sequence s.t.

$$|\lambda_n - rn^2| \leq Cn \qquad \forall\, n \geq 1.$$

Let $\kappa > \frac{3}{2}\pi^2$. Then

$$\sum_{n\geq 0} |a_n|^2 \ll \mathrm{e}^{2\kappa/(rT)} \int_{-T/2}^{T/2} \Big| \sum_{n\geq 0} a_n \mathrm{e}^{i\lambda_n t} \Big|^2 \mathrm{d}t \qquad \forall\, (a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{C}),$$

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Idea of the proof (I)

We construct a sequence $(g_n)_{n\in\mathbb{N}^*}$ of entire functions satisfying:

$$\text{(A1)}\quad |g_n(x)| \ll_\Lambda K_\tau/\{1+|x-\lambda_n|^2\} \qquad (x\in\mathbb{R});$$

$$\text{(A2)}\quad g_n(z) \ll_n \mathrm{e}^{\pi\tau|z|} \qquad (z\in\mathbb{C});$$

$$\text{(A3)}\quad g_n(\lambda_k) = \delta_{kn} \text{ for all } k,\, n\in\mathbb{N}^*.$$

By Paley–Wiener, g_n is, for $n\in\mathbb{N}^*$, the Fourier transform of $f_n\in L^2(\mathbb{R})$ supported in $[-\tau/2,\tau/2]$.
Moreover, the sequence $(t\mapsto f_k(t-\tau/2))_k$ is biorthogonal (in $L^2(0,\tau)$) to the sequence $\left(\mathrm{e}^{-2\pi i\lambda_k}\right)$. We continue as in the proof of Ingham's theorem.

Idea of the proof (II)

The main difficulty is the construction of the sequence (g_n). This is done in two steps:

1. Construction (via infinite products) of a sequence of entire functions (Ψ_n) such that $\Psi_n(\lambda_k) = \delta_{nk}$ and

$$\Psi_n(z) \ll \mathrm{e}^{\pi\sqrt{|z-\lambda_n|}}\{1 + |z - \lambda_n|\}^B.$$

2. Find the best "multiplier" " $H(z)$ such that $g_n(z) = \Psi_n(z)H(z)$ satisfies (A1).

How does number theory help

A Beurling type inequality in 2D

Theorem. (Tenenbaum et Tucsnak (2008))
For every $b, \tau > 0$, there exists $\delta(b, \tau) > 0$ s.t.

$$\int_0^\tau \int_0^b \Big| \sum_{m,n\in\mathbb{Z}} a_{mn} \mathrm{e}^{i\left(nx+\{m^2+n^2\}t\right)} \Big|^2 \mathrm{d}x\mathrm{d}t \geq \delta \sum_{m,n\in\mathbb{Z}} |a_{mn}|^2 \qquad (1)$$

for every $(a_{mn}) \in \ell^2(\mathbb{Z}\times\mathbb{Z}, \mathbb{C})$.

Remark. The result is obvios for $b,\ \tau \geq 2\pi$.

On the distribution of the sums of two squares

Lemma 1.
Let $\mathcal{N} = \{m^2 + n^2 \mid m, n \in \mathbb{Z}\}$. Then there exists an absolute constant $C > 0$ s.t. $\sup_{y\in\mathbb{R}} \left|\mathcal{N}\cap]y, y+x]\right| \leq C_3 \frac{x}{\sqrt{\ln x}} \quad (x \geq 2).$

Theorem (Selberg) Let M, $N \in \mathbb{N}$ and let $\mathcal{A} \subset]M, M+N] \cap \mathbb{N}$. Assume that, for each prime power p^r, $\mathcal{A}$ is excluded from $w(p^r)$ residue classes modulo p^r and, furthermore, that, for each p, the forbidden residue classes $\mod p^r$ and $\mod p^s$ are disjoint whenever $r \neq s$. Then, for each $Q > 1$, we have

$$|\mathcal{A}| \leq \frac{N + Q^2}{L}$$

with

$$L := \sum_{d\leq Q} \prod_{p^r \| d} \left\{\frac{1}{\vartheta(p^r)} - \frac{1}{\vartheta(p^{r-1})}\right\}, \quad \vartheta(p^r) := 1 - \sum_{1\leq s\leq r} \frac{w(p^s)}{p^s}.$$

A simple artihmetic lemma

Lemma. *Let* $n,\ x,\ y,\ z \in \mathbb{Z}$ *be such that*

$$n^2 + x^2 = (n+1)^2 + y^2 = (n+2)^2 + z^2.$$

Then n *is odd. In particular the system*

$$n^2 + x^2 = (n+1)^2 + y^2 = (n+2)^2 + z^2 = (n+3)^2 + w^2,$$

has no solutions $n,\ x,\ y,\ w \in \mathbb{Z}$.

Proof. Assume that n and x are even.
Then $1 + y^2 \equiv 0(4)$, which is a contradiction.
If n is even and x is odd, then $x^2 \equiv 1(8)$ and $(n+2)^2 - n^2 \equiv 4(8)$, so that

$$z^2 \equiv n^2 + x^2 - (n+2)^2 \equiv 1 - 4 \equiv 5(8),$$

which is impossible.

Consequence. (1) holds with $T \geq \frac{3}{2}$, e.g., $I_1 = I_2 = [0, \frac{3}{4}]$.

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On the distribution of lattice points on a circle

Lemma 2.
For $M, N, V \in \mathbb{N}^*$, we senote by $Z = Z(M,N,V)$ the set of $n \in \mathbb{N}$ s.t. $M < n \leq M + N$ and $V - n^2$ is a square. Then

$$|Z| \leq C\sqrt{N \log(2N)},$$

for some constant $C > 0$.

Proof. Use again Selbergs' sieve and ...
Remark. In the case of a rectangle with incommensurable side-lentghs we also need some diophantine approximation.

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