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Backward Stochastic Differential Equations

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1 Introduction

We shall discuss on the backward stochastic differential equation (BSDE) of the form

$$(\mathbf{P}) : \begin{cases} -dY_t + \partial\varphi(Y_t)dt \ni F(t, Y_t, Z_t)dt - Z_t dW_t, \\ Y_T = \xi, \quad t \in [0, T]. \end{cases}$$

When $\partial\varphi \neq \emptyset$, then $(\mathbf{P})$ is called backward stochastic variational inequality BSVI).

The first paper concerned with **BSDE**s:

Bismut, J.M. (**1973**) *Conjugate convex functions in optimal stochastic control.* J. Math. Anal. Appl.,44, p. 384-404.

He introduced a nonlinear Ricatti BSDE and showed the existence and uniqueness of bounded solutions.

Pardoux, E. and Peng, S. (**1990**) *Adapted solution of a backward stochastic differential equation.* Systems Control Lett., 14, p. 55-61. considered general BSDEs, and this paper was the starting point for the development of the study of these equations.

The interest in these equations is not confined to pure mathematicians - they have important applications in the theory of mathematical finance; in particular, they play a major role in hedging and nonlinear pricing theory for imperfect markets.

First, the theory of contingent claim valuation in a complete market studied by Black and Scholes (1973), Merton (1973, 1991), Karatzas (1989), among others, can be expressed in terms of BSDEs.

Indeed, the problem is to determine the price of a contingent claim $\xi \geq 0$ of maturity T , which is a contract that pays an amount ξ at time T . In a complete market it is possible to construct a portfolio which attains as final wealth the amount ξ. Thus, the dynamics of the value of the replicating portfolio Y are given by a BSDE with linear generator f, with Z corresponding to the hedging portfolio. Using BSDE theory, we will show there exist a unique price and a unique hedging portfolio —by restricting admissible strategies to square-integrable ones.

In certain applications the state Y_t should be maintained in a (convex) domain K. Practically this is realized with a supplementary drift $-\partial I_K(Y_t)$ in the equation. In this case instead of the above model, it is considered the model:

$$\begin{cases} -dY_t + \partial I_K(Y_t)dt \ni F(t, Y_t, Z_t)dt - Z_t dW_t, \\ Y_T = \xi \in K, \qquad t \in [0, T]. \end{cases}$$

or more general model:

- stochastic equations with a supplementary subdifferential drift

$$\begin{cases} -dY_t + \partial\varphi(Y_t)dt \ni F(t, Y_t, Z_t)dt - Z_t dW_t, \\ Y_T = \xi \in \overline{Dom\,(\varphi)}, \qquad t \in [0, T]. \end{cases} \tag{1}$$

  that is a BSVI.

Hence given a nonempty closed convex set K, a final (*maturity*) moment $T>0$ and a final value (*contingent claim*) $\xi \in K$, a supplementary source $-\partial I_K(Y_t)$ on the BSDE arrives to maintain the solution (*price*) $Y_s \in K$ for all $0 \leq s \leq T$.

It is naturally to put the question: given the equation

$$(\mathbf{P1}):\begin{cases} -dY_t = F(t,Y_t,Z_t)dt - Z_t dW_t, \\ Y_T = \xi, \qquad t \in [0,T]. \end{cases}$$

what are the conditions on the coefficient F such that the price Y_t satisfies the constrain $Y_s \in K$, for all $0 \leq s \leq T$.? This last problem is the **viability problem** of K for the BSDE $(P1)$.

We present an existence and uniqueness result for the backward stochastic variational inequalityBSVI) in Hilbert spaces :

$$(\mathbf{P}):\begin{cases} -dY_t + \partial\varphi(Y_t)dt \ni F(t,Y_t,Z_t)dt - Z_t dW_t, \\ Y_T = \xi, \qquad t \in [0,T]. \end{cases}$$

The first remark (and very important !) is on $Dom(\varphi)$:

- usually, in the case of progressively SVI, it is assumed

$$int(Dom(\varphi)) \neq \emptyset;$$

- in the case of BSVI it is not necessary to put this assumption.

Remark *In the ∞–dimensional case, the condition $int(Dom(\varphi)) \neq \emptyset$ is, in general, a very strong assumption.*

Remark from the beginning that the problem $(\mathbf{P})$ is a general approach for

- multivalued boundary Neumann backward stochastic problem:

$$(1):\begin{cases} -dY_t - \Delta Y_t dt = F(t, Y_t, Z_t)dt - Z_t dW_t), \\ \hfill \text{on } \Omega \times [0,T] \times D, \\ -\dfrac{\partial Y(t,x)}{\partial n} \in \partial j(Y(t,x)), \quad \text{on } \Omega \times]0,T[\times \Gamma, \\ Y(\omega, T, x) = \xi(\omega, x), \text{ on } \hfill \Omega \times D \end{cases}$$

In this case $\mathbb{H} = L^2(D)$, D is a bounded domain from $\mathbb{R}^d$ with $\Gamma = Bd\,(D)$ sufficiently smooth and $\varphi : \mathbb{H} \rightarrow]-\infty, +\infty]$ is given by

$$\varphi(u) = \begin{cases} \dfrac{1}{2}\int_D |grad\, u|^2 dx + \int_\Gamma j(u) d\sigma, \text{ if } u \in H^1(D) \text{ and } j(u) \in L^1(\Gamma), \\ +\infty, \hfill \text{otherwise.} \end{cases}$$

- multivalued Dirichlet backward stochastic problem:

$$(2):\begin{cases} -dY_t - \Delta Y_t dt + \partial j(Y_t)dt \ni F(t,Y_t,Z_t)dt - Z_t dW_t), \\ \hfill \text{on } \Omega\times[0,T]\times D, \\ Y(\omega,t,x)=0, \quad \text{on } \Omega\times[0,T]\times\Gamma, \\ Y(\omega,T,x)=\xi(\omega,x), \quad \text{on } \Omega\times D. \end{cases}$$

Now $\varphi:\mathbb{H}=L^2(D)\to]-\infty,+\infty]$ is given by

$$\varphi(u)=\begin{cases} \frac{1}{2}\int_D |grad\, u|^2dx+\int_D j(u(x))dx, & \text{if } u\in H^1(D) \text{ and } j(u)\in L^1(\Gamma), \\ +\infty, & \text{otherwise.} \end{cases}$$

- and the multivalued BSPDE coming from porous media models

$$(3):\begin{cases} -dY_t - \Delta\left(\partial j(Y_t)\right)dt \ni F(t,Y_t,Z_t)dt - Z_t dW_t, \\ \hfill \text{on } \Omega\times[0,T]\times D, \\ Y(\omega,T,x)=0, \quad \text{on } \Omega\times D \\ \partial j(Y(\omega,t,x)) \ni 0, \quad \text{on } \Omega\times]0,T[\times\Gamma \end{cases}$$

In this case $\varphi : \mathbb{H} = H^{-1}(D) \to]-\infty, +\infty]$

$$\varphi(u) = \begin{cases} \int_D j(u(x))dx, & \text{if } u \in L^1(D), j(u) \in L^1(D), \\ +\infty, & \text{otherwise.} \end{cases}$$

(The corresponding forward SDE for porous media was considered (2006) by V.Barbu - G.Da Prato - M. Rckner: *Existence of strong solutions for stochastic porous media equation* .

A second motivation of the study

In finite dimensional case if we consider the stochastic differential system

$$X_s^x = x + \int_0^s b(X_r^x)dr + \int_0^s \sigma(X_r^{t,x})dW_r,\ s \geq 0$$

and the scalar multivalued BSDE

$$Y_{t;s}^x + \int_s^t U_{t;r}^x\ dr = g(X_t^x) + \int_s^t f(X_r^x, Y_{t;r}^x, Z_{t;r}^x)dr - \int_s^t Z_{t;r}^x dW_r,\ s \in [0,t],$$
$$U_{t;r}^x \in \partial\varphi(Y_{t;r}^x)$$

($\varphi : \mathbb{R} \to]-\infty, +\infty]$ is a l.s.c. convex function), then

$$u\,(t,x) = Y_{t;0}^x$$

is a (viscosity) solution of the *parabolic variational inequality* (in particular a parabolic obstacle problem for $\partial\varphi = \partial I_K$)

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \mathcal{L}u(t,x) + \partial\varphi(u(t,x)) \ni f(x,u(t,x),\sigma^*(x)\nabla_x u_i(t,x)), \\ \hfill t \geq 0,\ x \in \mathbb{R}^m \\ u(0,x) = g(x), \quad x \in \mathbb{R}^m, \end{cases}$$

where the second order differential operator:

$$\mathcal{L}\psi(x) \overset{def}{=} \frac{1}{2}\sum_{j,\ell=1}^{m} (\sigma\sigma^*)_{jk}(x)\frac{\partial^2\psi(x)}{\partial x_j \partial x_k} + \sum_{j=1}^{m} b_j(x)\frac{\partial\psi(x)}{\partial x_j}, \quad \psi \in C^2(\mathbb{R}^m).$$

In infinite dimensional case if we denote

- $B_{pol}(\mathbb{H})$ the set of measurable functions $g : \mathbb{H} \to \mathbb{R}$ with polynomial growth,
- $\mathcal{L} : Dom(\mathcal{L}) \subset B_{pol}(\mathbb{H}) \to B_{pol}(\mathbb{H})$ the linear operator

$$\mathcal{L}\psi(x) = \frac{1}{2}\mathbf{Trace}\left[\sigma(x)\sigma^*(x)D^2\psi(x)\right] + \langle b(x), \nabla\psi(x)\rangle$$

- $P_t : B_{pol}(\mathbb{H}) \to B_{pol}(\mathbb{H})$ the transition semigroup:

$$P_t(g)(x) = \mathbb{E}g(X_t^x), \quad x \in \mathbb{H}, \ t \geq 0$$

Then

$$u(t,x) = P_t(g)(x)$$

is solution of the linear Kolmogorov equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \mathcal{L}u(t,x) = 0, & t > 0, \ x \in \mathbb{H} \\ u(0,x) = g(x), & x \in \mathbb{H} \end{cases}$$

(assuming $g \in C^2_{pol}(\mathbb{H})$).

Considering on Hilbert space BSDE in Hilbert of the form (P), with $\varphi = 0$, Fuhrman and Tessitore (2006) prove that

$$u(t,x) = Y^x_{t;0}$$

is solution of the nonlinear Kolmogorov equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \mathcal{L}u(t,x) = f(x, u(t,x), \sigma^*(x)\nabla_x u(t,x)), \\ \hfill t > 0,\ x \in \mathbb{H} \\ u(0,x) = g(x), \quad x \in \mathbb{H}, \end{cases}$$

Open problem !: Multivalued Kolmogorov problem in Hilbert space:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \mathcal{L}u(t,x) + \partial\varphi(u(t,x)) \ni f(x, u(t,x), \sigma^*(x)\nabla u(t,x)), \\ \hfill t > 0,\ x \in \mathbb{H} \\ u(0,x) = g(x), \quad x \in \mathbb{H}, \end{cases}$$

2 Martingale representation theorem

Denote $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ a complete right continuous stochastic bases. We will assume that

$$\mathcal{F}_t = \sigma(\{\beta_j(s),\ 0 \leq s \leq t,\ j \in I\!N^*\}) \vee \mathcal{N}$$

where $\mathcal{N}$ is the $\mathbb{P}$-null sets of $\mathcal{F}$ and $\{\beta_j, j \in \mathbb{N}^*\} \subset L^2(\Omega; C([0,T];\mathbb{R}))$ is a family of independent real valued standard Wiener processes;

If $(\mathbb{H}, |\cdot|)$ is a real separable Hilbert space we denote:

- $S^r_{\mathbb{H}}[0,T] = L^r_{ad}(\Omega; C([0,T];\mathbb{H})) \subset L^r(\Omega, \mathcal{F}, \mathbb{P}; C([0,T];\mathbb{H}))$, $r \geq 1$, the closed linear subspace of continuous adapted stochastic processes,;
- $L^r_{ad}(\Omega; L^q(0,T;\mathbb{H})) \subset L^r(\Omega; L^q(0,T;\mathbb{H}))$, $r, q \geq 1$, the closed linear subspace of progressively measurable processes

- $\mathcal{M}^r(\Omega \times [0,T];\mathbb{H})$ the space of continuous r-martingales M, that is:

$$\begin{array}{ll} m_1) & M \text{ is a continuous adapted stochastic process,} \\ m_2) & M_0(\omega) = 0, \quad a.s.\ \omega \in \Omega, \\ m_3) & \mathbb{E}\,|M_t|^r < \infty, \ \ \forall\, t \geq 0, \\ m_4) & \mathbb{E}(M_t|\mathcal{F}_s) = M_s, \quad \text{if } s \leq t. \end{array}$$

  If $r > 1$ then

$$\mathcal{M}^r(\Omega \times [0,T];\mathbb{H}) \subset S^r_{\mathbb{H}}[0,T], \ \text{ is a closed linear subspace.}$$

Let

$$\mathbb{K}_0 \subset \mathbb{K} \subset \tilde{\mathbb{K}}$$

three real separable Hilbert spaces such that

- $$\mathbb{K}_0 = Q^{1/2}(\mathbb{K})$$

  where $Q : \mathbb{K} \to \mathbb{K}$ is a linear bounded self–adjoint strictly positive operator; and

- the embedding $J : \mathbb{K}_0 \to \tilde{\mathbb{K}}$ is Hilbert–Schmidt.

If$\{g_j\}$ is an orthonormal complete bases in $\mathbb{K}_0$ then

$$W_t = \sum_j g_j \beta_j(t), \quad t \geq 0,$$

defines a Wiener process on $\tilde{\mathbb{K}}$ (a cylindrical Wiener process on $\mathbb{K}$) with JJ^* the covariance operator.

- Denote $L_2^0 = L_2(\mathbb{K}_0, \mathbb{H})$ the separable Hilbert space of all Hilbert–Schmidt operators $U \in L(\mathbb{K}_0, \mathbb{H})$ that is

$$\|U\|^2 = \sum_i |U g_i|^2 < \infty$$

- Denote $\Lambda^p_{\mathbb{H}\times\mathbb{K}_0}(0,T)$, $\quad p \in [0,\infty[$, the space of progressively measurable processes $Z:\Omega\times]0,T[\to L_2^0$ such that:

$$\|Z\|_p = \begin{cases} \left[\mathbb{E}\left(\int_0^T \|Z_s\|_Q^2 ds\right)^{\frac{p}{2}}\right]^{\frac{1}{p}\wedge 1} < \infty, & \text{if } p > 0, \\ \mathbb{E}\left[1 \wedge \left(\int_0^T \|Z_s\|_Q^2 ds\right)^{\frac{1}{2}}\right], & \text{if } p = 0. \end{cases}$$

The space $(\Lambda^p_{\mathbb{H}\times\mathbb{K}_0}(0,T), \|\cdot\|_p)$, $p\geq 1$, is a Banach space and $\Lambda^p_{\mathbb{H}\times\mathbb{K}_0}(0,T)$, $0 \leq p < 1$, is a complete metric space with the metric $\rho(Z_1,Z_2)=\|Z_1-Z_2\|_p$; when $p = 0$ the metric convergence coincide with the probability convergence.

The stochastic integral

$$I(Z)(t) = \int_0^t Z_s dW_s$$

is defined as linear continuous operator

$$I : \Lambda^p_{\mathbb{H}\times\mathbb{K}_0}(0,T) \longrightarrow S^p_{\mathbb{H}}[0,T] \stackrel{def}{=} L^p_{ad}(\Omega; C([0,T];\mathbb{H})$$

for $p \in [0,\infty[$, and has the properties:

$$\begin{aligned}
&(a) \qquad \mathbb{E} I(Z)(t) = 0, \quad \text{if } p \geq 1,\\
&(b) \qquad \mathbb{E}|I(Z)(T)|^2 = \|Z\|_2^2, \quad \text{if } p \geq 2,\\
&(c) \qquad \frac{1}{c_p}\|Z\|_p^p \leq \mathbb{E} \sup_{t\in[0,T]} |I(Z)(t)|^p \leq c_p \|Z\|_p^p,\\
&\qquad\qquad \text{if } p > 0 \quad (\text{Burkholder-Davis-Gundy inequality})\\
&(d) \qquad I(Z) \in \mathcal{M}^p(\Omega\times[0,T];\mathbb{H})
\end{aligned}$$

Theorem 1. (Martingale representation)

(*a*) *If* $T > 0$ *and* $\xi \in L^p(\Omega, F_T, \mathbb{P}; \mathbb{H})$, $p > 1$, *then there exists a unique* $Z \in \Lambda_W^p$ *such that*

$$\xi = \mathbb{E}\xi + \int_0^T Z_s dW_s \tag{2}$$

(*b*) *If* $M \in \mathcal{M}^p(\Omega \times [0,T]; \mathbb{H})$, *then there exists a unique* $Z \in \Lambda_W^p$ *such that*

$$M(t) = \int_0^t Z_s dW_s. \tag{3}$$

Proof. b) The representation result (b) follows from (a) for $\xi = M(T)$ and passing to conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_t)$.

a) *Uniqueness.* If Z_1, Z_2 satisfy (2), then

$$\|Z_1 - Z_2\|_2^2 \leq \mathbb{E}\left|\int_0^T Z_1(s)dW_s - \int_0^T Z_2(s)dW_s\right|^2 = 0,$$

which yields $Z_1 = Z_2$.

Existence. One proves, for $p = 2$, that

$$\mathcal{L} = \left\{ h + \int_0^T Z_s dW_s : \; h \in \mathbb{H}, \; Z \in \Lambda^2_W \right\}$$

is a closed linear subspace dense in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{H})$.

Proposition 2. (Itô's formula) *Let*

$$\begin{aligned} i) \qquad & \xi \in L^0(\Omega, F_T, \mathbb{P}; \mathbb{H}), \\ ii) \qquad & F \in L^0_{ad}(\Omega; L^1(0, T; \mathbb{H})) \\ iii) \qquad & Z \in \Lambda^0_{\mathbb{H} \times \mathbb{K}_0}(0, T), \\ iv) \qquad & Y \in L^0_{ad}(\Omega; C([0, T]; \mathbb{H})), \end{aligned}$$

such that

$$Y_t = \xi + \int_t^T F(s) ds - \int_t^T Z_s dW_s, \; \forall\, t \in [0, T], \mathbb{P} - a.s. \tag{1.9}$$

If $\psi : [0,T] \times \mathbb{H} \to R$ *and its derivatives* $\psi'_t, \psi'_x, \psi''_{xx}$ *are uniformly continuous on bounded subsets on* $[0,T] \times \mathbb{H}$*, then* $\mathbb{P}$*-a.s., for all* $t \in [0,T]$:

$$\psi(t, Y_t) + \int_t^T \left[\psi'_t(s, Y_s) + \frac{1}{2}\mathbf{Tr}(\psi''_{xx}(s, Y_s) Z_s Q Z^*(s) \right] ds$$
$$= \psi(T, \xi) + \int_t^T \langle \psi'_x(s, Y_s, F(s)) \rangle \, ds - \int_t^T \langle \psi'_x(s, Y_s), Z_s dW_s \rangle \, .$$

In particular, for $\psi(t,x) = |x|^2$ *we have:*

Energy Equality

$$|Y_t|^2 + \int_t^T \|Z_s\|^2 ds = |\xi|^2 + 2\int_t^T \langle Y_s, F(s) \rangle \, ds - 2\int_t^T \langle Y_s, Z_s dW_s \rangle \, ,$$

$\forall t \in [0,T], \ \mathbb{P} - a.s. \ \omega \in \Omega.$

3 Backward stochastic differential equation (BSDE)

3.1 BSDE: Lipschitz condition

Consider the following BSDE

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{4}$$

where we assume

◊ $p > 1$,

$$\xi \in L^p\left(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{H}\right) \tag{5}$$

◊ *the function* $F(\cdot, \cdot, y, z) : \Omega \times [0, T] \to \mathbb{H}$ *is* $\mathcal{P}$*–measurable for every* $(y, z) \in \mathbb{H} \times L_2(\mathbb{K}_0, \mathbb{H})$,

$\diamondsuit$ *there exist* $L \in L^1(0,T)$, $\ell \in L^2(0,T)$ *such that*

$$\begin{cases} (I) \quad \textit{Lipschitz conditions:} \\ \textit{for all } y, y' \in \mathbb{H},\ z, z' \in L_2(\mathbb{K}_0, \mathbb{H}),\ d\mathbb{P} \otimes dt - a.e.\ : \\ \qquad (L_y) \qquad |F(t,y',z) - F(t,y,z)| \le L(t)\,|y'-y|\ , \\ \qquad (L_z) \qquad |F(t,y,z') - F(t,y,z)| \le \ell(t)\,|z'-z|\ ; \\ (II) \quad \textit{Boundedness condition:} \\ \qquad (B_F) \qquad \mathbb{E}\left(\int_0^T |F(t,0,0)|\,dt\right)^p < \infty\ . \end{cases} \tag{6}$$

We recall the notation

$$S_{\mathbb{H}}^{1+}[0,T] \overset{def}{=} \bigcup_{p>1} S_{\mathbb{H}}^{p}[0,T]$$

Theorem 1 *Let* $p > 1$ *and the assumptions (5) and (6) be satisfied. Then the BSDE (4) has a unique solution* $(Y,Z) \in S_{\mathbb{H}}^p[0,T] \times \Lambda_{\mathbb{H}\times\mathbb{K}_0}^p(0,T)$. *Moreover uniqueness holds in* $S_{\mathbb{H}}^{1+}[0,T] \times \Lambda_{\mathbb{H}\times\mathbb{K}_0}^0(0,T)$.

Proof.

The existence and uniqueness of the solution (Y,Z) is obtained by Banach fixed point theorem in the Banach space $S_{\mathbb{H}}^{p}[0,T]\times\Lambda_{\mathbb{H}\times\mathbb{K}_0}^{p}(0,T)$ equipped with an equivalent norm

$$\|(Y,Z)\|_{(N,p)}^{p}\overset{def}{=}\max_{i=\overline{1,N}}\ \mathbb{E}\left[\left(\sup_{r\in[T_{i-1},T_i]}|Y_r|^p\right)+\left(\int_{T_{i-1}}^{T_i}|Z_r|^2\,dr\right)^{p/2}\right],$$

and $T_i=\dfrac{iT}{N}$. The mapping

$$\Phi:S_{\mathbb{H}}^{p}[0,T]\times\Lambda_{\mathbb{H}\times\mathbb{K}_0}^{p}(0,T)\longrightarrow S_{\mathbb{H}}^{p}[0,T]\times\Lambda_{\mathbb{H}\times\mathbb{K}_0}^{p}(0,T),$$

$\Phi(U,V)=(Y,Z)$, defined by

$$\begin{aligned}&Y_t=\mathbb{E}(\xi+\textstyle\int_t^T F(s,U_s,V_s)ds|\mathcal{F}_t),\\&\xi+\textstyle\int_0^T F(s,U_s,V_s)ds=\mathbb{E}\left[\xi+\textstyle\int_0^T F(s,U_s,V_s)ds\right]+\displaystyle\int_0^T Z_s dW_s,\end{aligned}$$

is a contraction for N large enough

$\blacksquare$

3.2 BSDE: Monotone case.

We shall assume :

$(\mathbf{BSDE} - \mathbf{MH}_F)$:

♦ *$\xi : \Omega \to R^m$ is a $\mathcal{F}_T$ − measurable random vector;*

♦ *the function $F(\cdot, \cdot, y, z) : \Omega \times [0, T] \to \mathbb{R}^m$ is $\mathcal{P}$−measurable for every $(y, z) \in \mathbb{R}^m \times \mathbb{R}^{m \times k}$,*

♦ *there exist some deterministic functions $\mu \in L^1(0, T; \mathbb{R})$ and $\ell \in L^2(0, T; \mathbb{R})$ such that*

$$
\left|
\begin{array}{ll}
(I) \ \text{for all } y,y' \in \mathbb{R}^m,\ z,z' \in \mathbb{R}^{m\times k},\ d\mathbb{P}\otimes dt - a.e.\ : & \\
& \textit{Continuity:} \\
(C_y) & y \longrightarrow F(t,y,z) : \mathbb{R}^m \to \mathbb{R}^m \ \textit{is continuous}\ ; \\
& \textit{Monotonicity condition:} \\
(M_y) & \langle y'-y, F(t,y',z)-F(t,y,z)\rangle \le \mu(t)\,|y'-y|^2\ ; \\
& \textit{Lipschitz condition:} \\
(L_z) & |F(t,y,z')-F(t,y,z)| \le \ell(t)\ |z'-z|\ ; \\
(II) \ \textit{Boundedness condition:} & \\
(B_F) & \displaystyle\int_0^T F_\rho^{\#}(t)\,dt < \infty,\ \ a.s., \quad \forall\, \rho \ge 0.
\end{array}
\right. \tag{7}
$$

where

$$
F_\rho^{\#}(t) = \sup\left\{|F(t,y,0)| : |y| \le \rho\right\}.
$$

Theorem 2. *Let $p > 1$ and the assumptions* $(\mathbf{BSDE} - \mathbf{MH}_F)$ *be satisfied. If for all* $\rho \geq 0$:

$$\mathbb{E}\ |\xi|^p + \mathbb{E}\ \left(\int_0^T F_\rho^{\#}(t)dt \right)^p < \infty,$$

then the BSDE:

$$Y_t = \xi + \int_t^T F\left(s, Y_s, Z_s\right) ds - \int_t^T Z_s dB_s, \quad a.s.$$

has a unique solution $(Y, Z) \in S_{\mathbb{H}}^p [0,T] \times \Lambda_{\mathbb{H}\times\mathbb{K}_0}^p (0,T)$. *Moreover, uniqueness holds in* $S_{\mathbb{H}}^{1+} [0,T] \times \Lambda_{\mathbb{H}\times\mathbb{K}_0}^0 (0,T)$, *where*

$$S_{\mathbb{H}}^{1+} [0,T] \overset{def}{=} \bigcup_{p>1} S_{\mathbb{H}}^p [0,T].$$

Proof.

Step 1. Uniqueness.

Step 2. *Existence* is obtained passing to limit in the approximating equation

$$Y_\varepsilon(t) = \xi + \int_t^T F_\varepsilon(s, Y_\varepsilon, Z_\varepsilon)ds - \int_t^T Z_\varepsilon(s)dW_s \tag{8}$$

where

$$F_\varepsilon(t, y, z) = \alpha\Gamma_\varepsilon(t, y, z) + \frac{1}{\varepsilon}\left(\Gamma_\varepsilon(t, y, z) - y\right) = F(t, \Gamma_\varepsilon(t, y, z), z)$$

and Γ_ε is the unique solution of the equation

$$\Gamma_\varepsilon + \varepsilon\left[\alpha\Gamma_\varepsilon - F(t, \Gamma_\varepsilon, z)\right] = y.$$

3.3 BSDE - multivalued monotone case (BSVI)

Consider the BSVI

$$\begin{cases} -dY_t + \partial\varphi(Y_t)dt \ni F(t, Y_t, Z_t)dt - Z_t dW_t, \\ Y_T = \xi, \qquad t \in [0, T], \end{cases} \tag{9}$$

where F satisfies $(\mathbf{BSDE} - \mathbf{MH}_F)$ and

$$\varphi : \mathbb{H} \longrightarrow]-\infty, +\infty], \text{ is a proper convex s.c.i. function} \tag{10}$$

$\partial\varphi$ denotes the subdifferential :

$$\partial\varphi(u) = \{h \in \mathbb{H} : \langle h, v - u\rangle + \varphi(u) \leq \varphi(v), \ \forall v \in \mathbb{H}\},$$
$$Dom\,(\partial\varphi) = \{u \in \mathbb{H} : \partial\varphi(u) \neq \emptyset\}.$$

Definition. *The solution of BSVI (9) is a couple* $(Y,Z) \in S^0_{\mathbb{H}}[0,T] \times \Lambda^0_{\mathbb{H}\times\mathbb{K}_0}(0,T)$ *such that*

$$\left\{\begin{array}{ll} a) & Y(\omega,t) \in Dom\,(\partial\varphi), (\omega,t) - a.e., \\ b) & \exists\, U \in L^0_{ad}(\Omega \times L^1(0,T;\mathbb{H})), \\ & \qquad U(\omega,t) \in \partial\varphi(Y(\omega,t)), (\omega,t) - a.e., \end{array}\right.$$

and

$$Y_t + \int_t^T U(s)ds = \xi + \int_t^T F(s,Y_s,Z_s)ds - \int_t^T Z_s dW_s,$$

$\forall\ t \in [0,T],\ a.s.\ \omega \in \Omega.$

We add the following assumptions for F

(A) *There exist* $p \geq 2$, *a positive stochastic process* $\beta \in L^1\left(\Omega \times \left]0,T\right[\right)$, *a positive function* $b \in L^1(0,T)$ *and a real number* $\kappa \geq 0$, *such that*

$$
\begin{array}{ll}
(i) & \mathbb{E}\varphi^+(\xi) < \infty, \quad \textit{and} \\
(ii) & \textit{for all } (u,\hat{u}) \in \partial\varphi \textit{ and } z \in R^{m\times k}: \\
& \quad \langle \hat{u}, F(t,u,z) \rangle \leq \frac{1}{2}|\hat{u}|^2 + \beta_t + b(t)|u|^p + \kappa|z|^2 \\
& \qquad d\mathbb{P}\otimes dt - a.e.\ (\omega,t) \in \Omega \times [0,T]
\end{array}
\tag{11}
$$

Theorem 3. *Let $p \geq 2$ and the assumptions* $(\mathbf{BSDE} - \mathbf{MH}_F)$ *be satisfied. Suppose moreover that for all* $\rho \geq 0$

$$\mathbb{E}\ |\xi|^p + \mathbb{E}\ \left(\int_0^T F_\rho^{\#}(s)ds\right)^p < \infty \tag{12}$$

and the assumption $(\mathbf{A})$ *is satisfied. Then there exists a unique pair* $(Y, Z) \in S_{\mathbb{H}}^p [0, T] \times \Lambda_{\mathbb{H}\times\mathbb{K}_0}^p (0, T)$ *and a unique stochastic process* $U \in \Lambda_{\mathbb{H}\times\mathbb{K}_0}^2 (0, T)$ *such that*

$$(a) \qquad \int_0^T |F(t, Y_t, Z_t)|\, dt < \infty, \ \ \mathbb{P} - a.s.,$$

$$(b) \qquad Y_t(\omega) \in Dom(\partial\varphi), \quad d\mathbb{P} \otimes dt \ - \ a.e. \ (\omega, t) \in \Omega \times [0, T],$$

$$(c) \qquad U_t(\omega) \in \partial\varphi(Y_t(\omega)), \quad d\mathbb{P} \otimes dt \ - \ a.e. \ (\omega, t) \in \Omega \times [0, T]$$

and for all $t \in [0, T]$:

$$Y_t + \int_t^T U_s ds = \xi + \int_t^T \Phi(s, Y_s, Z_s)\, ds - \int_t^T Z_s dB_s, \ a.s. \tag{13}$$

Moreover, uniqueness holds in $S_{\mathbb{H}}^{1+}[0,T] \times \Lambda_{\mathbb{H}\times\mathbb{K}_0}^{0}(0,T)$.

Proof.

We have to prove only the existence.

Let $\varepsilon \in]0,1]$. Consider the approximating equation

$$Y_t^\varepsilon + \int_t^T \nabla\varphi_\varepsilon(Y_s^\varepsilon)ds = \xi + \int_t^T F(s, Y_s^\varepsilon, Z_s^\varepsilon)ds - \int_t^T Z_s^\varepsilon dW_s, \tag{14}$$

where

$$\begin{aligned}\varphi_\varepsilon(u) &= \inf\left\{\frac{1}{2\varepsilon}|v-u|^2 + \varphi(v) : v\in\mathbb{H}\right\} \\ &= \frac{1}{2\varepsilon}|u - J_\varepsilon u|^2 + \varphi(J_\varepsilon u)\end{aligned}$$

and

$$J_\varepsilon u = (I + \varepsilon\partial\varphi)^{-1}(u), \quad \nabla\varphi_\varepsilon(u) = \frac{1}{\varepsilon}(u - J_\varepsilon u)$$

By Theorem 2 the equation (14) has a unique solution $(Y^{\varepsilon}, Z^{\varepsilon}) \in S_{\mathbb{H}}^{p}[0,T] \times \Lambda_{\mathbb{H}\times\mathbb{K}_0}^{p}(0,T)$ and moreover

$$\mathbb{E}^{\mathcal{F}_t} \sup_{s\in[t,T]} |Y_s^{\varepsilon}|^p + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T \varphi_{\varepsilon}(Y_s^{\varepsilon})ds\right)^{p/2} + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T |Z_s^{\varepsilon}|^2 ds\right)^{p/2} \leq C\ \mathbb{E}^{\mathcal{F}_t}\ \left[|\eta|^p + \left(\int_t^T |F(s,u_0,0)|\,ds\right)^p\right],$$

and

$$\mathbb{E}\int_0^T |\nabla\varphi_{\varepsilon}(Y_s^{\varepsilon})|^2 ds \leq C \tag{15}$$

To obtain (15-d) we apply the

Lemma 7. (Stochastic subdifferential inequality) *Let* $\psi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ *be a function of class* C^1 *such that for all* $t \geq 0$ $\psi(t,\cdot) : \mathbb{R}^d \to \mathbb{R}$ *is a convex function. Let* $\{X_t\,;\, t \geq 0\}$ *be a continuous semimartingale. Then* $\mathbb{P}-a.s.$ *for* *all* $t \leq T$:

$$\psi(t, X_t) + \int_t^T \left[\frac{\partial\psi(r, X_r)}{\partial r} dr + \langle \nabla_x \psi(r, X_r),\, dX_r \rangle\right] \leq \psi(T, X_T) \tag{16}$$

Proof.

Let $t = t_0 < t_1 < t_2 < \cdots < t_n = T$ such that $t_{i+1} - t_i = (T - t)/n$. For each $0 \leq i < n$, from the the definition of the subdifferential operator, it follows that

$$\psi(t_i, X_{t_i}) + \left[\psi(t_{i+1}, X_{t_{i+1}}) - \psi(t_i, X_{t_{i+1}})\right] + \left\langle \nabla_x \psi(t_i, X_{t_i}),\, X_{t_{i+1}} - X_{t_i} \right\rangle \leq \psi(t_{i+1}, X_{t_{i+1}}).$$

The result follows summing over i and by taking the limit $n \to \infty$.

■

Hence

$$\varphi_\varepsilon(Y_t^\varepsilon) + \int_t^T \langle \nabla\varphi_\varepsilon(Y_s^\varepsilon)), dY_s^\varepsilon \rangle \leq \varphi_\varepsilon(\xi)$$

and consequently

$$\varphi_\varepsilon(Y_t^\varepsilon) + \int_t^T |\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2 ds \le \varphi(\xi) + \int_t^T \langle \nabla\varphi_\varepsilon(Y_s^\varepsilon), F(s, Y_s^\varepsilon, Z_s^\varepsilon)\rangle\, ds$$
$$- \int_t^T \langle \nabla\varphi_\varepsilon(Y_s^\varepsilon), Z_s^\varepsilon dW_s\rangle$$

that yields (15).

Now using the inequality

$$\langle y - \tilde{y}, \nabla\varphi_\varepsilon(y) - \nabla\varphi_\delta(\tilde{y})\rangle \ge -(\varepsilon + \delta)\langle \nabla\varphi_\varepsilon(y), \nabla\varphi_\delta(\tilde{y})\rangle$$

we obtain by Energy Equality for $Y_\varepsilon - Y_\delta$ that

$$\mathbb{E}(\sup_{t\in[0,T]} |Y_\varepsilon(t) - Y_\delta(t)|^2) + |||Z_\varepsilon - Z_\delta|||_2^2 \le C(\varepsilon + \delta) \tag{17}$$

■

4 Viability for BSDE and PDE

Consider the semiliniar parabolic system

$$\begin{cases} 1 \leq i \leq n \\ -\dfrac{\partial u_i(t,x)}{\partial t} - \mathcal{A}(t)u_i(t,x) = f_i(t,x,u(t,x),\sigma^*(t,x)\nabla_x u_i(t,x)),\ (t,x) \in \left]0,T\right[\times \mathbb{R}^d \\ u(T,x) = H(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{18}$$

where $\mathcal{A}(t)$ is a second differential operator

$$\begin{aligned} \mathcal{A}(t)\varphi(x) &= \frac{1}{2}\mathbf{Tr}[\sigma\sigma^*(t,x)D^2_{xx}\varphi(x)] + \langle b(t,x), \nabla_x\varphi(x)\rangle \\ &= \frac{1}{2}\sum_{j,\ell=1}^{d}(\sigma\sigma^*)_{j\ell}(t,x)\frac{\partial^2\varphi(x)}{\partial x_j \partial x_\ell} + \sum_{j=1}^{d} b_j(t,x)\frac{\partial\varphi(x)}{\partial x_j}\,, \quad \varphi \in C^2\left(\mathbb{R}^d\right), \end{aligned}$$

and

$$
\begin{aligned}
b &: [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d, \\
\sigma &: [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times k}, \\
H &: \mathbb{R}^d \longrightarrow \mathbb{R}^n, \\
f_i &: [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R},\ 1 \leq i \leq n
\end{aligned}
$$

are continuous and moreover

$$
\begin{aligned}
|b\left(t,x\right) - b\left(t,\tilde{x}\right)| + |\sigma\left(t,x\right) - \sigma\left(t,\tilde{x}\right)| &\leq L\left|x-\tilde{x}\right|, \\
|f\left(t,x,y,z\right) - f\left(t,x,\tilde{y},\tilde{z}\right)| &\leq L\left|y-\tilde{y}\right| + L\left|z-\tilde{z}\right|
\end{aligned}
$$

4.1 Viscosity solutions

M. CRANDALL, H. ISHII, P.L. LIONS: *User's Guide to the Viscosity Solutions of Second Order Partial Differential Equations*, Bul. AMS 27, 1-67, 1992.

Let $\mathbb{S}^{d\times d} \subset \mathbb{R}^{d\times d}$ the set of nonnegative definite symmetric matrix $d \times d$.
Let $u \in C((0,T) \times \mathbb{R}^d)$ and $(t,x) \in (0,T) \times \mathbb{R}^d$.
Definition

(a) $(p,q,S) \in D^{1,2+}u(t,x) \subset \mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^{d\times d}$ *((p,q,S) is a parabolic superjet of function u in the point (t,x)) if*

$$p(s-t) + \langle q, y-x\rangle + \frac{1}{2}\langle S(y-x), y-x\rangle \geq [u(s,y) - u(t,x)]$$
$$+o\left(|s-t| + |y-x|^2\right), \qquad \forall\ (s,y) \in \mathcal{V}_{(t,x)}$$

or equivalent

$$\liminf_{|s-t|+|y-x|^2\to 0} \frac{\left[u(t,x) + p(s-t) + \langle q, y-x\rangle + \frac{1}{2}\langle S(y-x), y-x\rangle\right] - u(s,y)}{|s-t| + |y-x|^2} \geq 0$$

(b) $(p,q,S) \in D^{1,2-}u(t,x) \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^{d\times d}$ *((p,q,S) is a parabolic subjet of the function u in the point (t,x)) if*

$$p(s-t) + \langle q, y-x\rangle + \frac{1}{2}\langle S(y-x), y-x\rangle \leq [u(s,y) - u(t,x)]$$
$$+o\left(|s-t| + |y-x|^2\right), \qquad \forall\ (s,y) \in \mathcal{V}_{(t,x)}$$

or equivalent

$$\limsup_{|s-t|+|y-x|^2 \to 0} \frac{\left[u(t,x) + p(s-t) + \langle q, y-x\rangle + \frac{1}{2}\langle S(y-x), y-x\rangle\right] - u(s,y)}{|s-t| + |y-x|^2} \leq 0$$

We now give the definition of the viscosity solution of PDE (18):

Definition

(a) $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *is a u.s.c viscosity subsolution of PDE (18) if*

$$u(T,x) \leq g(x),\ \ \forall\ x \in \mathbb{R}^d$$

and $\forall\ (t,x)\in(0,T)\times\mathbb{R}^d,\ \ \forall\ (p,q,S)\in D^{1,2+}u(t,x):$

$$-p-\frac{1}{2}\mathbf{Tr}\left(\sigma(t,x)\sigma^*(t,x)S\right)-\langle b(t,x),q\rangle\leq f(t,x,u(t,x),\sigma^*(t,x)q)$$

(b) $\quad u:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ *is a l.s.c viscosity supersolution of PDE (18) if*

$$u(T,x)\geq g(x),\ \ \forall\ x\in\mathbb{R}^d$$

and $\forall\ (t,x)\in(0,T)\times\mathbb{R}^d,\ \ \forall\ (p,q,S)\in D^{1,2-}u(t,x):$

$$-p-\frac{1}{2}\mathbf{Tr}\left(\sigma(t,x)\sigma^*(t,x)S\right)-\langle b(t,x),q\rangle\geq f(t,x,u(t,x),\sigma^*(t,x)q)$$

(c) $\quad u\in C([0,T]\times\mathbb{R}^d)$ *is a viscosity solution of PDE (18) if* u *is sub and super - solution of PDE (18).*

Let $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t,W_t)_{t\geq0}$ a Wiener process $W:\Omega\times[0,\infty[\ \to\mathbb{R}^k$, $(t,x)\in[0,T]\times\mathbb{R}^d$.

The SDE:

$$\begin{cases} X_s^{t,x} = x, & s \in [0,t], \\ dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dW_s, & s \in]t,T]. \end{cases} \tag{19}$$

has a unique solution

$$X_\cdot^{t,x} \in L_{ad}^p \left(\Omega; C\left[0,T\right]; \mathbb{R}^d\right), \ \ \forall\ p \geq 1.$$

The BSDE

$$\begin{cases} Y_s^{t,x} = H(X_{\tilde{T}}^{t,x}), & s \in \left[\tilde{T}, T\right] \\ Y_s^{t,x} = H(X_{\tilde{T}}^{t,x}) + \int_s^{\tilde{T}} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^{\tilde{T}} Z_r^{t,x} dW_r, & s \in \left[t, \tilde{T}\right] \\ Y_s^{t,x} = Y_t^{t,x}, & s \in [0,t]. \end{cases} \tag{20}$$

has a unique solution

$$\left(Y_\cdot^{t,x}, Z_\cdot^{t,x}\right) \in L_{ad}^p \left(\Omega; C\left[0,T\right]; \mathbb{R}^d\right) \times L_{ad}^p \left(\Omega; L^2\left(0,T; \mathbb{R}^{d\times k}\right)\right)$$

Moreover

$$u\left(t,x\right)=Y_t^{t,x}$$

is a viscosity solution of partial differential system

$$\begin{cases} -\dfrac{\partial u_i(t,x)}{\partial t}-\mathcal{A}(t)u_i(t,x)=f_i(t,x,u(t,x),\sigma^*(t,x)\nabla_x u_i(t,x)), \\ u(T,x)=H(x), \qquad (t,x)\in[0,T]\times\mathbb{R}^d,\ 1\le i\le n, \end{cases}$$

and

$$Y_s^{t,x}=u\left(s,X_s^{t,x}\right)$$

5 Viability

Let

$$\mathcal{K}=\{K(t,x)=\overline{K(t,x)}\subset\mathbb{R}^n:(t,x)\in[0,T]\times\mathbb{R}^d\}.$$

Definition

(a) *The BSDE (20) is K–viable on* $[0,T]$ *if* $\forall\ (t,x) \in [0,T] \times \mathbb{R}^d,\ \forall\ \tilde{T} \in [t,T],\ H \in C_{pol}(\mathbb{R}^d, \mathbb{R}^n)$ *such that*

$$H(\tilde{x}) \in K(\tilde{T}, \tilde{x}),\ \forall\ \tilde{x} \in \mathbb{R}^d$$

it follows

$$Y_s^{t,x} \in K(s, X_s^{t,x}), \quad \forall\ s \in [t, \tilde{T}],\ \mathbb{P} - a.s.$$

(b) *The PDE (18) is K–viable on* $[0,T]$ *if* $\forall\ \tilde{T} \in [0,T], \forall\ H \in C_{pol}(\mathbb{R}^d, \mathbb{R}^n)$ *such that*

$$H(x) \in K(\tilde{T}, x),\ \forall\ x \in \mathbb{R}^d,$$

$\exists\ u \in C_{pol}([0,\tilde{T}] \times \mathbb{R}^d, \mathbb{R}^n),\ u(\tilde{T}, x) = H(x)$ for all $x \in \mathbb{R}^d$, *a viscosity solution of PDE*:

$$-\frac{\partial u_i(t,x)}{\partial t} - \mathcal{A}(t) u_i(t,x) = f_i(t, x, u(t,x), \sigma^*(t,x) \nabla_x u_i(t,x)),$$

$(t,x) \in [0, \tilde{T}[\times \mathbb{R}^d,\ 1 \le i \le n,$ *such that*

$$u(t,x) \in K(t,x),\ \forall (t,x) \in [0,\tilde{T}] \times \mathbb{R}^d.$$

Remark . *Since* $Y_s^{t,x} = u\left(s, X_s^{t,x}\right)$ *it follows that the BSDE (20) is* $K-$ *viable on* $[0,T]$ *iff the PDE (18) is* K*–viable on* $[0,T]$.

Theorem (Viability criterion for BSDE and PDE)

Assume

(i) $\quad (t,x) \mapsto d^2_{K(t,x)}(y) : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ *is u.s.c.,*

(ii) $\quad \exists \;\; M > 0,\ p \geq 1$ *such that*

$$d^2_{K(t,x)}(0) \leq M(1 + |x|^p),\ \forall (t,x) \in [0,T] \times \mathbb{R}^d.$$

Then the following assertions are equivalent:

(c) *The BSDE (**??**) (and respectively the PDE (18)) is* K*–viable on* $[0,T]$.

(cc) $\exists \;\; C > 0$ *such that for all* $0 \leq t \leq s \leq \tilde{T}$

$$d^2_{K(s,X_s^{t,x})}(Y_s^{t,x;\tilde{T}}) \leq e^{C\left(\tilde{T}-s\right)} \mathbb{E}^{\mathcal{F}_s}\left[d^2_{K(\tilde{T},X_{\tilde{T}}^{t,x})}(H(X_{\tilde{T}}^{t,x}))\right],\ \mathbb{P}-a.s.$$

(ccc) $\exists \;\; C > 0$ *such that* $\forall\ z \in \mathbb{R}^{n \times d}$*, the function* $h(t,x,y) = d^2_{K(t,x)}(y)$ *is* *a viscosity subsolution of PDE*

$$-\frac{\partial V(t,x,y)}{\partial t}-\mathcal{L}_z(t)V(t,x,y)+\langle \nabla_y V(t,x,y), f(t,x,y,\sigma^*(t,x)z^*)\rangle = Cd^2_{K(t,x)}(y),$$

$(t,x,y) \in \,]0,T[\, \times \mathbb{R}^d \times \mathbb{R}^n$, *where*

$$\mathcal{L}_z(t)\varphi(x,y) = \frac{1}{2}\mathbf{Tr}\Big[\sigma(t,x)\sigma^*(t,x)[\varphi''_{xx}(x,y) + z^*\varphi''_{yx}(x,y) + \varphi''_{xy}(x,y)\, z \\ + z^*\varphi''_{yy}(x,y)\, z]\Big] + \langle b(t,x), \varphi'_x(x,y)\rangle$$

Corollary (Viability criterion for BSDE and PDE)
Let $K(t,x) \equiv K(t)$, $(t,x) \in [0,T] \times \mathbb{R}^d$. *Then the following assertions are equivalent:*
(j) *The BSDE (**??**) (and respectively the PDE (18)) is K–viable on* $[0,T]$.
(jj) $\exists\ C > 0$ *such that* $\forall\ z \in \mathbb{R}^{n\times d}$, *the function* $V(t,y) = d^2_{K(t)}(y)$ *is a*

viscosity subsolution of PDE

$$-\frac{\partial V(t,y)}{\partial t} - \mathcal{A}_z\,(t;x)\,V(t,y) + \langle \nabla_y V(t,x,y), f\,(t,x,y,\sigma^*(t,x)z^*)\rangle = Cd^2_{K(t)}(y),$$
$$(t,y) \in \,]0,T[\, \times \mathbb{R}^n,$$

where

$$\mathcal{A}_z\,(t;x)\,\psi(y) = \frac{1}{2}\mathrm{Tr}[\sigma(t,x)\sigma^*(t,x)\ z^*\psi''_{yy}(y)z],$$

$i.e.$ $\forall\ (t,y) \in (0,T) \times \mathbb{R}^n,\ \ \forall\ (p,q,S) \in D^{1,2+}V(t,y)$:

$$-p - \frac{1}{2}\mathbf{Tr}\,(\sigma(t,x)\sigma^*(t,x)z^*Sz) + \langle q, f\,(t,x,y,\sigma^*(t,x)z^*)\rangle \le Cd^2_{K(t)}(y),\ \ \forall\, x \in \mathbb{R}^d$$

Example.

Let $a \in C^2([0,T];\mathbb{R}^d)$ and $r \in C^2([0,T];\mathbb{R}_+)$ *such that* $r\,(t) \ge \delta > 0$, $t \in [0,T]$. Let

$$\mathcal{K} = \{K\,(t) : t \ge 0\}$$

$$\begin{aligned} K(t) &= \overline{B\,(a\,(t)\,, r(t))} \\ &= \{x \in \mathbb{R}^n : |x - a\,(t)| \le r(t)\}, \quad t \ge 0. \end{aligned}$$

Then the PDE (18)) is $\mathcal{K}$–viable on $[0,T]$ iff $\forall\ (t,x,z) \in [0,T] \times \mathbb{R}^d \ \times \mathbb{R}^{n\times k}$, $\forall v \in \mathbb{R}^n$ such that

$$|v| = 1 \quad \text{and} \quad \sigma^*(t,x)z^*v = 0$$

if follows

$$r'\left(t\right) + \left\langle v, a'\left(t\right)\right\rangle + \left\langle v, f(t,x,a\left(t\right) + r\left(t\right)v, \sigma^*(t,x)z^*)\right\rangle \leq \frac{1}{2r\left(t\right)}\left\|z\sigma\left(t,x\right)\right\|^2$$

or equivalent $\forall y \in \mathbb{R}^n$ *such that*

$$|y - a\left(t\right)| = r\left(t\right) \quad \text{and} \quad \sigma^*(t,x)z^*\left(y - a\left(t\right)\right) = 0$$

it follows

$$r'\left(t\right)r\left(t\right) + \left\langle y - a\left(t\right), a'\left(t\right)\right\rangle + \left\langle y - a\left(t\right), f(t,x,y,\sigma^*(t,x)z^*)\right\rangle \leq \frac{1}{2}\left\|z\sigma\left(t,x\right)\right\|^2$$

Thank you for your attention !