FLAG PARAPRODUCTS

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(I) Classical paraproducts

- kernel representation

$$T(f_1,\dots,f_m)(x) = \int_{\mathbb{R}^m} f_1(x-t_1)\dots f_m(x-t_m)\, K(t)\, dt$$

where K is a CZ kernel in $\mathbb{R}^m$.

- Multiplier representation

$$T(f_1,\dots,f_m)(x) = \int_{\mathbb{R}^m} m(\xi)\, \hat{f}_1(\xi_1)\dots \hat{f}_m(\xi_m)\, e^{2\pi i x(\xi_1+\dots+\xi_m)}\, d\xi$$

where $m \in L^\infty(\mathbb{R}^m)$ satisfies

$$|\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^{|\alpha|}}$$

for many multi-indices α.

- Coifman-Meyer theorem

$T : L^{p_1} \times \dots \times L^{p_m} \to L^p$ as long as

$1 < p_1, \dots, p_m \leq \infty$, $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$, $0 < p < \infty$

- Kenig - Stein, Grafakos Torres

- A bit about the proof

Write $K(t) = \sum_{k\in\mathbb{Z}} \Phi_k^1(t_1)\cdot\ldots\cdot\Phi_k^m(t_m)$.

then, $T(f_1,\ldots,f_m) = \sum_{k\in\mathbb{Z}} (f_1 * \Phi_k^1)\ldots(f_m * \Phi_k^m)$ and

$$\|T(f_1,\ldots,f_m)\|_p = \left|\int_{\mathbb{R}} T(f_1,\ldots,f_m)(x)\, f_{m+1}(x)\, dx\right| =$$

$$= \left|\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} (f_1 * \Phi_k^1)(x)\ldots(f_m * \Phi_k^m)(x)(f_{m+1} * \Phi_k^{m+1})(x)\, dx\right|$$

$$\overset{\text{easy}}{\leq} \int_{\mathbb{R}} \Big(\sum_{k\in\mathbb{Z}} |f_1 * \Phi_k^1|^2\Big)^{1/2} \Big(\sum_{k\in\mathbb{Z}} |f_2 * \Phi_k^2|^2\Big)^{1/2} \cdot \prod_{j\neq 1,2} \sup_k |f_j * \Phi_k^j|\, dx$$

$$= \int_{\mathbb{R}} S(f_1)(x)\cdot S(f_2)(x)\cdot \prod_{j\neq 1,2} M(f_j)(x)\, dx$$

when S is the Littlewood - Paley square function and M is the Hardy - Littlewood maximal function.

- This argument "proves" the "Banach - case"
- The general theorem follows by using CZ decompositions for each of the functions $f_1,\ldots,f_m$ carefully.

I) Flag paraproducts

- Kernel representations

$$T(f,g,h)(x) = \int_{\mathbb{R}^7} f(x-\alpha_1-\beta_1)\, g(x-\alpha_2-\beta_2-\gamma_1)\, h(x-\alpha_3-\gamma_2)\, K(\alpha) K(\beta) K(\gamma)\, d\alpha\, d\beta\, d\gamma$$

where $K(\beta)$, $K(\gamma)$ are CZ kernels in $\mathbb{R}^2$ and $K(\alpha)$ is a CZ kernel in $\mathbb{R}^3$.

- Multiplier representation

$$T(f,g,h)(x) = \int_{\mathbb{R}^3} m(\xi)\, \hat{f}(\xi_1)\, \hat{g}(\xi_2)\, \hat{h}(\xi_3)\, e^{2\pi i x(\xi_1+\xi_2+\xi_3)}\, d\xi$$

where $m(\xi) = \tilde{m}(\xi_1,\xi_2,\xi_3) \cdot \tilde{\tilde{m}}(\xi_1,\xi_2) \cdot \tilde{\tilde{\tilde{m}}}(\xi_2,\xi_3)$

Theorem (M, Revista Iber. 2007)

$T_{ab} : L^{p_1} \times L^{p_2} \times L^{p_3} \to L^p$ for any $1 < p_1, p_2, p_3 < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p}$, $0 < p < \infty$ where

$$T_{ab}(f,g,h)(x) = \int_{\mathbb{R}^3} a(\xi_1,\xi_2)\, b(\xi_2,\xi_3)\, \hat{f}(\xi_1)\, \hat{g}(\xi_2)\, \hat{h}(\xi_3)\, e^{2\pi i x(\xi_1+\xi_2+\xi_3)}\, d\xi$$

- A bit about the proof

Decompose the kernels $K(\alpha)$, $K(\beta)$, $K(\gamma)$ as before

$$K(\alpha) = \sum_{k_1} \tilde{\Phi}_{k_1}(\alpha_1)\,\tilde{\tilde{\Phi}}_{k_1}(\alpha_2)\,\tilde{\tilde{\tilde{\Phi}}}_{k_1}(\alpha_3),$$

$$K(\beta) = \sum_{k_2} \tilde{\Phi}_{k_2}(\beta_1)\,\tilde{\tilde{\Phi}}_{k_2}(\beta_2),$$

$$K(\gamma) = \sum_{k_3} \tilde{\Phi}_{k_3}(\gamma_1)\,\tilde{\tilde{\Phi}}_{k_3}(\gamma_2).$$

then, $T(f,g,h) =$

$$\sum_{k_1,k_2,k_3} \left(f*\tilde{\Phi}_{k_1}*\tilde{\Phi}_{k_2}\right)\left(g*\tilde{\tilde{\Phi}}_{k_1}*\tilde{\tilde{\Phi}}_{k_2}*\tilde{\Phi}_{k_3}\right)\left(h*\tilde{\tilde{\tilde{\Phi}}}_{k_1}*\tilde{\tilde{\Phi}}_{k_3}\right).$$

- there are no "easy estimates" this time
- CZ decomposition is also ineffective due to the "product structure" of the kernel.

(III) Why should one care ?

1. General Leibnitz rules
2. NLS (Germain, Masmoudi, Shatah, work in progress

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(1) Aims to understand how to estimate generic expressions such as

$$\left\| D^{\gamma}\left[D^{\alpha}(f_1\cdot f_2)\cdot D^{\beta}(f_4\cdot f_5\cdot f_6\cdot f_7)\right]\right\|_r \ ?$$

- Recall the Leibnitz rule:

$$\|D^\alpha(fg)\|_p \lesssim \|D^\alpha f\|_{p_1} \cdot \|g\|_{q_1} + \|f\|_{p_2} \cdot \|D^\alpha g\|_{q_2}$$

for any $1 < p_i, q_i \leq \infty$, $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{p}$, $0 < p < \infty$

Proof: Use Littlewood-Paley dec. & paraproduct theory:

- $f = \sum_{k \in \mathbb{Z}} f * \psi_k$, $g = \sum_{k \in \mathbb{Z}} g * \psi_k$. So,

$$f \cdot g = \sum_{k_1, k_2} (f * \psi_{k_1})(g * \psi_{k_2}) = \sum_{k_1 \sim k_2} + \sum_{k_1 \ll k_2} + \sum_{k_2 \ll k_1}$$

$= \mathrm{I} + \mathrm{II} + \mathrm{III}$.

- cd II for instance:

$$\mathrm{II} = \sum_{k_1 \ll k_2} (f * \psi_{k_1})(g * \psi_{k_2}) = \sum_{k_2} \Big(\sum_{k_1 \ll k_2} f * \psi_{k_1} \Big)(g * \psi_{k_2})$$

$$= \sum_{k_2} (f * \varphi_{k_2})(g * \psi_{k_2}) = \sum_k (f * \varphi_k)(g * \psi_k) =$$

$$= \sum_k \left[(f * \varphi_k)(g * \psi_k) \right] * \psi_k$$ for a well chosen and new ψ_k.

- Denote by

$$\boxed{\Pi(f, g) = \sum_k \left[(f * \varphi_k)(g * \psi_k) \right] * \psi_k}$$

So, $D^\alpha(\Pi(f,g)) = \sum_k [(f*\varphi_k)(g*\psi_k)] * D^\alpha \psi_k =$

$= \sum_k [(f*\varphi_k)(g*\psi_k)] * 2^{k\alpha} \tilde{\psi}_k =$

$= \sum_k [(f*\varphi_k)(g*2^{k\alpha}\psi_k)] * \tilde{\psi}_k =$

$= \sum_k [(f*\varphi_k)(g*D^\alpha \tilde{\tilde{\psi}}_k)] * \tilde{\psi}_k =$

$= \sum_k [(f*\varphi_k)(D^\alpha g * \tilde{\tilde{\psi}}_k)] * \tilde{\psi}_k :=$

$= \tilde{\Pi}(f, D^\alpha g).$

So, $\boxed{D^\alpha(\Pi(f,g)) = \tilde{\Pi}(f, D^\alpha g)}$

. Now, any $\Pi(f,g)$ can be rewritten as

$$\Pi(f,g)(x) = \int_{\mathbb{R}^2} m(\xi_1,\xi_2)\hat{f}(\xi_1)\hat{g}(\xi_2)e^{2\pi i x(\xi_1+\xi_2)}\,d\xi_1 d\xi_2$$

where $m(\xi_1,\xi_2) = \sum_k \hat{\varphi}_k(\xi_1)\hat{\psi}_k(\xi_2)\hat{\psi}_k(\xi_1+\xi_2)$

is a classical symbol.

then, Coifman - Meyer theorem solves the problem.

Want to estimate now

$\|D^\beta[h \cdot D^\alpha(fg)]\|_p$ - non-linearity of complexity 2 !

- Need to understand the correct way to "malify it".
- First, how should one malify $\underline{h \cdot \Pi(f,g)}$?
- Say $\Pi(f,g) = \sum_{k_1 \ll k_2} (f * \psi_{k_1})(g * \psi_{k_2})$
- Write $h = \sum_{k_3 \in \mathbb{Z}} h * \psi_{k_3}$ as before. Then,

$$h \cdot \Pi(f,g) = \sum_{k_1 \ll k_2, k_3} (f * \psi_{k_1})(g * \psi_{k_2})(h * \psi_{k_3}) =$$

$$= \sum_{\substack{k_1 \ll k_2 \\ k_3 \ll k_2}} + \sum_{\substack{k_1 \ll k_2 \\ k_3 \sim k_2}} + \sum_{k_1 \ll k_2 \ll k_3} .$$

- Observe that the first two expressions are just regular paraproducts, while the third one $\underline{\text{is not}}$. It can be written as:

$$\int_{\mathbb{R}^3} m(\xi_1, \xi_2, \xi_3) \widehat{f}(\xi_1) \widehat{g}(\xi_2) \widehat{h}(\xi_3) e^{2\pi i x(\xi_1 + \xi_2 + \xi_3)} d\xi$$

where $m(\xi_1, \xi_2, \xi_3) = \sum_{k_1 \ll k_2 \ll k_3} \widehat{\psi_{k_1}}(\xi_1) \widehat{\psi_{k_2}}(\xi_2) \widehat{\psi_{k_3}}(\xi_3) =$

$$= \sum_{k_2 \ll k_3} \widehat{\varphi_{k_2}}(\xi_1) \widehat{\psi_{k_2}}(\xi_2) \widehat{\psi_{k_3}}(\xi_3) =$$

$$= \sum_{k_2 \ll k_3} \widehat{\varphi_{k_2}}(\xi_1) \widehat{\psi_{k_2}}(\xi_2) \underline{\underline{\widehat{\varphi_{k_3}}(\xi_2)}} \cdot \widehat{\psi_{k_3}}(\xi_3) \stackrel{!}{=}$$

$$= \left(\sum_{k_2} \widehat{\varphi}_{k_2}(\xi_1)\widehat{\psi}_{k_2}(\xi_2)\right)\cdot\left(\sum_{k_3} \widehat{\varphi}_{k_3}(\xi_2)\cdot\widehat{\psi}_{k_3}(\xi_3)\right) =$$

$$= \underline{\underline{m(\xi_1,\xi_2)\cdot\widetilde{m}(\xi_2,\xi_3)}}.$$

- So the third term is a <u>flag paraproduct</u>, which we denote by $\Pi_{flag}(f,g,h)$.

- Observe also that

$$D^\beta\left(\Pi_{flag}(f,g,h)\right) =$$

$$= \int_{\mathbb{R}^3} |\xi_1+\xi_2+\xi_3|^\beta\, m(\xi_1,\xi_2)\cdot\widetilde{m}(\xi_2,\xi_3)\,\widehat{f}(\xi_1)\widehat{g}(\xi_2)\widehat{h}(\xi_3)\ldots$$

$$= \int_{\mathbb{R}^3} \frac{|\xi_1+\xi_2+\xi_3|^\beta}{|\xi_1|^\beta+|\xi_2|^\beta+|\xi_3|^\beta}\left(|\xi_1|^\beta+|\xi_2|^\beta+|\xi_3|^\beta\right)\cdot\ldots\ldots$$

$$:= \widetilde{\Pi}_{flag}(D^\beta f, g, h) + \widetilde{\Pi}_{flag}(f, D^\beta g, h) +$$

$+ \widetilde{\Pi}_{flag}(f, g, D^\beta h)$ where $\widetilde{\Pi}_{flag}$ is the 3-linear operator with symbol

$$\frac{|\xi_1+\xi_2+\xi_3|^\beta}{|\xi_1|^\beta+|\xi_2|^\beta+|\xi_3|^\beta}\cdot m(\xi_1,\xi_2)\cdot\widetilde{m}(\xi_2,\xi_3),$$ which is

a "flag symbol" if $\beta = 2, 4, \ldots$ etc.

this way, one reduces the study of the general Leibnitz rules to the study of flag paraproducts.

② Quadratic NLS

$$\begin{cases} \partial_t u + i\Delta u = u^2 & (t,x) \in \mathbb{R}\times\mathbb{R}^n \\ u|_{t=0} = u_0 \end{cases}$$

- Duhamel formula:

$$\boxed{\hat{u}(t,\xi) = \hat{u}_0(\xi)\cdot e^{it\xi^2} + \int_0^t e^{-i(s-t)\xi^2}\,\widehat{u^2}(s,\xi)\,ds}$$

- Write u as $\boxed{u = e^{-it\Delta} f}$. then Duhamel becomes:

$$\boxed{\hat{f}(t,\xi) = \hat{u}_0(\xi) + \int_0^t\int_{\mathbb{R}} e^{is\left(-\xi^2+\eta^2+(\xi-\eta)^2\right)}\hat{f}(s,\eta)\hat{f}(s,\xi-\eta)\,ds\,d\eta}$$

- Idea: Want to take advantage of the oscillation of the term $e^{is\Phi}$ when $\Phi := -\xi^2+\eta^2+(\xi-\eta)^2$.

- Problem: Φ is "too degenerate" (i.e $\Phi \equiv 0$ when $\xi=\eta$ or $\eta=0$) and so the usual $\frac{d}{ds}\left\{\frac{e^{is\Phi}}{i\Phi}\right\}$ - argument doesn't work.

- Need a "wiser integration by parts".

Denote by $\boxed{P := -\eta + \frac{1}{2}\xi}$ and

$$\boxed{Z := \phi + P\cdot(\partial_\eta \phi)}$$

- Observe that $Z = -\left(\eta^2 + (\xi+\eta)^2\right)$ which $= 0$ only at the origin.
- Alternatively, one has

$$\boxed{\frac{1}{iZ}\left(\partial_s + \frac{P}{s}\partial_\eta\right)e^{is\phi} = e^{is\phi}}$$

- In particular, RHS (Duhamel for f) =

$$= \int_0^t \int_{\mathbb{R}} \frac{1}{iZ}\left(\partial_s + \frac{P}{s}\partial_\eta\right)\left[e^{is\phi}\right] \hat{f}(\xi-\eta)\hat{f}(\eta)\,ds\,d\eta =$$

$$= \quad I + II.$$

- Using the fact that "$\frac{\xi^2}{iZ}$" & "$\frac{\eta^2}{iZ}$" are "classical symbols", Coifman-Meyer theorem proves that both (I) & (II) are "smoothing expressions". (decay in s also!)
- So, expressions of type

$$\boxed{\mathcal{F}^{-1}\int_0^t\int_{\mathbb{R}} e^{is\phi}\, m(\xi,\eta)\,\hat{g}(\eta)\,\hat{h}(\xi-\eta)\,ds\,d\eta}$$

appear naturally.

Also, one of the expressions related to $\underline{\underline{I}}$ is of the form

$$\mathcal{F}^{-1}\int_0^t\int_{\mathbb{R}} e^{is\Phi}\, m(\xi,\eta)\, \underline{\partial_s \hat{f}(\eta)}\, \hat{f}(\xi-\eta)\, ds\, d\eta \, . \qquad (*)$$

Since $u = e^{-is\Delta} f \Rightarrow f = e^{is\Delta} u \Rightarrow \dots$

$$\Rightarrow \quad \partial_s f = e^{is\Delta} u^2 \Rightarrow \underline{\widehat{\partial_s f}(\eta)} = e^{-is\eta^2}\, \widehat{u^2}(\eta)$$

$$= e^{-is\eta^2} \int_{\eta_1+\eta_2=\eta} \hat{u}(\eta_1)\cdot \hat{u}(\eta_2)\, d\eta_1\, d\eta_2 =$$

$$= e^{-is\eta^2} \int_{\eta_1+\eta_2=\eta} e^{+is\eta_1^2}\, \hat{f}(\eta_1)\, e^{+is\eta_2^2}\, \hat{f}(\eta_2)\, d\eta_1\, d\eta_2$$

$$= e^{-is\eta^2} \int_{\mathbb{R}} e^{+is\tau^2}\, \hat{f}(\tau)\, e^{+is(\eta-\tau)^2}\, \hat{f}(\eta-\tau)\, d\tau.$$

- Using this, $(*)$ becomes:

$$\mathcal{F}^{-1}\int_0^t\int_{\mathbb{R}^2} e^{is\tilde{\Phi}}\, m(\xi,\eta)\, \hat{f}(\tau)\, \hat{f}(\eta-\tau)\, \hat{f}(\xi-\eta)\, d\tau\, d\eta\, ds$$

where $\tilde{\Phi} = -\xi^2 + \cancel{\eta^2} + (\xi-\eta)^2 + (\eta-\tau)^2 + \tau^2$

- Using an "integration by parts argument" similar to the one before, one bumps into expressions of type:

$$\mathcal{F}^{-1}\int_0^t\int_{\mathbb{R}^2} e^{is\tilde{\Phi}}\,\underline{m(\xi,\eta)\,m(\xi,\eta,\tau)}\,\hat{f}(\tau)\hat{f}(\eta-\tau)\hat{f}(\xi-\eta)\,d\tau\,d\eta\,ds.$$

- Note that the inner formula is a <u>flag paraproduct</u>!
- So this time, the theorem about such tri-linear operators proves that such expressions are "smoothing".
- In the <u>3D case</u> <u>paraproduct</u> theory seems to be enough while <u>flag-paraproducts</u> are needed in the <u>2D case</u>.