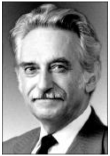
Modelling solid deforming bodies by using rate-type constitutive equations

Cristian Făciu

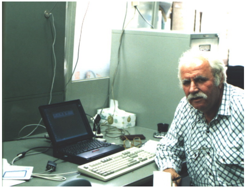
Institutul de Matematica ”Simion Stoilow” al Academiei Române
Mecanica mediilor continue

Milestones:

- Gr. C. Moisil
  - 1929, La mécanique analytique des systèmes continus, Thèse.
  - 1956, Shock waves in a cable. 9th Intern. Congr. Appl. Mech.



- N. Cristescu
  - Dynamic Plasticity, 1967, North Holland.
  (new version 2007, World Scientific, NJ)
- N. Cristescu and I. Suliciu
  - 1982, Viscoplasticiy, Martinus Nijhoff.



- I. Suliciu
  - 1979, An analogy between the constitutive equations of electric lines and those of 1D plasticity.
  - 1981, (with M. Mihăilescu-Suliciu) A rate type constitutive equation for the description of the corona effect, IEEE Trans.

Mechanical theories - application to wave propagation

Examples: - 1D isothermal case

Bar theory

unknowns: v, σ, ε

$$\begin{cases} \frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial X} = 0 \\ \frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial X} = 0 \\ \text{constitutive relation} \quad \sigma \leftrightarrow \varepsilon \end{cases}$$

Extensible string theory

unknowns: $\overrightarrow{v}$, $\overrightarrow{\lambda}$, T

$$\begin{cases} \frac{\partial \overrightarrow{v}}{\partial t} - \frac{\partial}{\partial X}\left(\frac{T}{\lambda}\overrightarrow{\lambda}\right) = 0 \\ \frac{\partial \overrightarrow{\lambda}}{\partial t} - \frac{\partial \overrightarrow{v}}{\partial X} = 0 \\ \text{constitutive relation} \quad T \leftrightarrow \lambda \end{cases}$$

$\sigma = \sigma_{eq}(\varepsilon)$ $\longleftrightarrow$ Monotone (convex or non-convex) elasticity

$\frac{\partial \sigma}{\partial t} = E(\varepsilon, \sigma, sign(\dot{\varepsilon}))\frac{\partial \varepsilon}{\partial t}$ $\longleftrightarrow$ Rate independent plasticity

$\frac{\partial \sigma}{\partial t} = E(\varepsilon, \sigma)\frac{\partial \varepsilon}{\partial t} + G(\varepsilon, \sigma)$ $\longleftrightarrow$ Viscoplasticity/Viscoelasticity

Conditions: $\sigma'_{eq}(\varepsilon) > 0$, $E(\varepsilon, \sigma, sign(\dot{\varepsilon})) > 0$, $E(\varepsilon, \sigma) > 0$

Mathematical theories - hyperbolic PDEs systems

System of conservation laws - elastic theory

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial X} = 0, \qquad \mathbf{U} \in \mathcal{D} \subset \mathbb{R}^n, \quad \mathbf{F} : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$$

or system with source term - viscoelastic/viscoplastic theory

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial X} = \mathbf{b}(\mathbf{U}), \quad \mathbf{A} : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad \mathbf{b} : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$$

$\frac{\partial \mathbf{F}}{\partial \mathbf{U}}$, or $\mathbf{A}(\mathbf{U})$ has real eigenvalues $\alpha_j(\mathbf{U})$ and n linear independent eigenvectors $r_j(\mathbf{U})$, j=1,n.

$\frac{dX}{dt} = \alpha_j(\mathbf{U}) \to$ wave speed

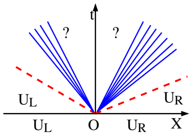
$(\alpha_j(\mathbf{U}), \mathbf{r}_j(\mathbf{U}))$ is "genuinely nonlinear" $\Longleftrightarrow \nabla \alpha_j(\mathbf{U}) \cdot r_j(\mathbf{U}) \neq 0$

$(\alpha_j(\mathbf{U}), \mathbf{r}_j(\mathbf{U}))$ is "linearly degenerate" $\Longleftrightarrow \nabla \alpha_j(\mathbf{U}) \cdot \mathbf{r}_j(\mathbf{U}) \equiv 0$

Problems, concepts, tools

Riemann problem: find weak solution for initial data

$$\mathbf{U}(X,0)=\begin{cases}\mathbf{U}_L, & \text{for } X<0\\ \mathbf{U}_R, & \text{for } X>0\end{cases}$$



Rarefaction wave: - integral curve of the vector-field r_j

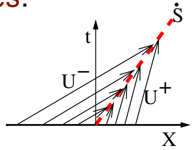
(solutions $\mathbf{U}(X,t)=\mathbf{U}(\xi)$, where $\xi=\frac{X}{t}\Longrightarrow \mathbf{U}'(\xi)=\mathbf{r}_j(\mathbf{U}(\xi))$)

Shock wave - a curve $X=\hat{X}(t)$ across which at least one of the components U_j has jump

Rankine-Hugoniot equations: $[\mathbf{U}]\,\dot{S}-[\mathbf{F}(\mathbf{U})]=0$ where $\dot{S}=\frac{\mathrm{d}\hat{X}}{\mathrm{d}t}$.

Uniqueness $\leftrightarrow$ Entropy condition on discontinuities:

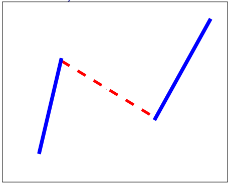
- Lax shock's inequalities, viscosity criteria,...



Numerical solutions

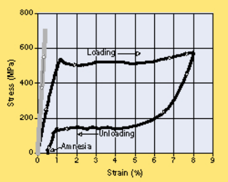
Phase transitions in solids → shape memory alloys

I. Suliciu, Mihaela Suliciu, C. F.: IJSS, 1987, IJES, 1990, 1992; Scripta Mater., 1994, Eur. J. Solid. Mech., 1996



physical assumption

$\sigma = \sigma_{eq}(\varepsilon)$ non-monotone ←



$$\left\{ \begin{array}{l} \frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial X} = 0 \\ \frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial X} = 0 \\ \sigma = \sigma_{eq}(\varepsilon) \end{array} \right.$$

⟶

⟵ $0 \leftarrow \mu$

$$\left\{ \begin{array}{l} \frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial X} = 0 \\ \frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial X} = 0 \\ \frac{\partial \sigma}{\partial t} - E\frac{\partial \varepsilon}{\partial t} = -\frac{E}{\mu}(\sigma - \sigma_{eq}(\varepsilon)) \end{array} \right.$$

Characteristics: $\frac{\mathrm{d}X}{\mathrm{d}t} = \pm\sqrt{\frac{\mathrm{d}\sigma_{eq}(\varepsilon)}{\mathrm{d}\varepsilon}}$

Characteristics: $\frac{\mathrm{d}X}{\mathrm{d}t} = \pm\sqrt{E}$, $\frac{\mathrm{d}X}{\mathrm{d}t} = 0$

$E > 0$, $\mu > 0$

↓

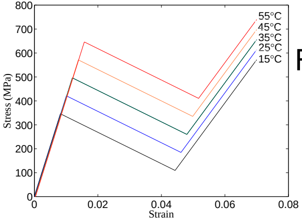
Hyperbolic-elliptic system ! (ill posed problems)

↓

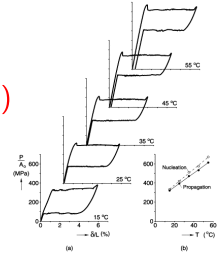
Hyperbolic system - irrespective of the slope

Thermal effects →C. F., M. Suliciu, Int. J. Solids Structures, 2002

Pseudo-elastic response: NiTi alloy



Fit of $\sigma = \sigma_{eq}(\varepsilon, \theta)$ ←



thermoelastic approach

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial \sigma}{\partial X} = 0 \\ \dfrac{\partial \varepsilon}{\partial t} - \dfrac{\partial v}{\partial X} = 0 \\ C\dfrac{\partial e}{\partial t} - \sigma\dfrac{\partial \varepsilon}{\partial t} = \kappa\dfrac{\partial^2 \theta}{\partial X^2} - \omega(\theta - \theta_0) \\ \sigma = \sigma_{eq}(\varepsilon, \theta) \\ e = e_{eq}(\varepsilon, \theta) \end{cases}$$

rate-type approach

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial \sigma}{\partial X} = 0 \\ \dfrac{\partial \varepsilon}{\partial t} - \dfrac{\partial v}{\partial X} = 0 \\ \tilde{C}\dfrac{\partial e}{\partial t} - \sigma\dfrac{\partial \varepsilon}{\partial t} = \kappa\dfrac{\partial^2 \theta}{\partial X^2} - \omega(\theta - \theta_0) \\ \dfrac{\partial \sigma}{\partial t} - E\dfrac{\partial \varepsilon}{\partial t} = -\dfrac{E}{\mu}(\sigma - \sigma_{eq}(\varepsilon, \theta)) \\ e = e_M(\varepsilon, \sigma, \theta) \end{cases}$$

$$! \quad e_M(\varepsilon, \sigma_{eq}(\varepsilon, \theta), \theta) = e_{eq}(\varepsilon, \theta)$$

Modelling quasistatic response of SMA alloy

Equilibrium state: $(\varepsilon_0, \sigma_{eq}(\varepsilon_0, \theta_0), \theta_0)$

$\downarrow$

Perturbation of an equilibrium state

$\downarrow$

Linearized system

$\downarrow$

Fourier's method:

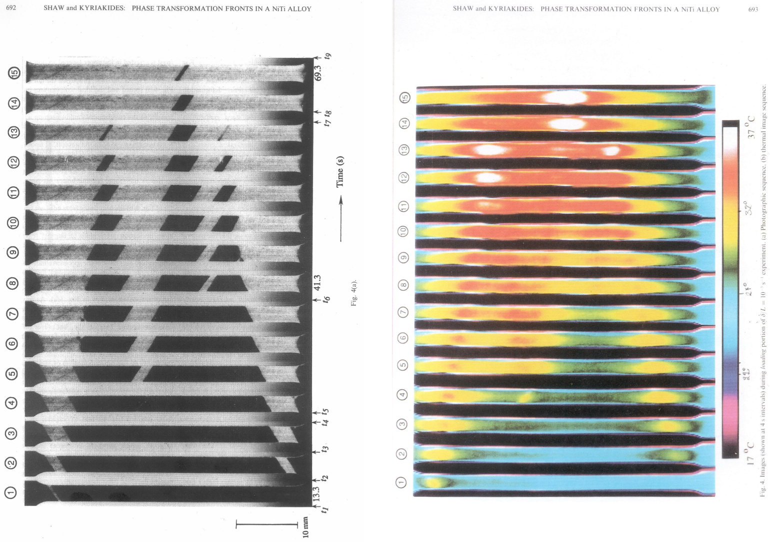
$v(X,t) = \exp(rt)\sin(X\sqrt{\beta})$, $\varepsilon(X,t) = \exp(rt)\cos(X\sqrt{\beta})$, $\cdots$

$\downarrow$

If $E_0 = \sigma'_{eq}(\varepsilon_0) > 0 \Rightarrow$ stability phenomena $\leftrightarrow r \approx \frac{-E_0}{\mu} < 0$.

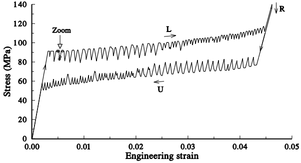
If $E_0 = \sigma'_{eq}(\varepsilon_0) < 0 \Rightarrow$ instability phenomena $\leftrightarrow r \approx \frac{-E_0}{\mu} > 0$.

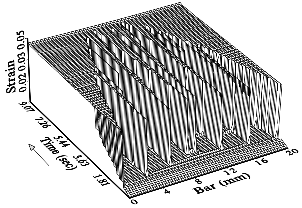
Thermo-mechanical events. (Experiment Shaw & Kyriakides, 1997)

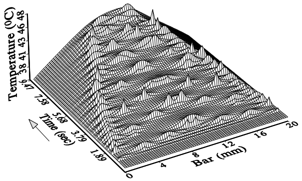


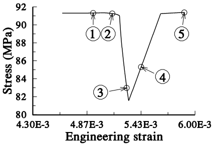
Full field monitoring: deformation and temperature during A→M phase transformation.

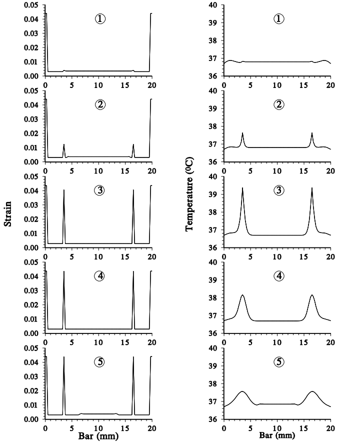
C. F. & M. Suliciu, Int. J. Solids Structures, 2002











Modelling dynamic response of SMA alloy

C. F. & A. Molinari, Int. J. Solids Structures, 2006

Impact problems

Elastic system	Rate-type system
$\sigma = \sigma_{eq}(\varepsilon)$	$\dot{\sigma} - E\dot{\varepsilon} = -\frac{E}{\mu}\lvert\sigma - \sigma_{eq}(\varepsilon)\rvert^{\lambda-1}(\sigma - \sigma_{eq}(\varepsilon))$
$\downarrow$	$\downarrow$
Exact (weak) solution	**Numerical solution**
Jump relations - sharp interfaces -	**Traveling waves - smooth interfaces -**

$$\begin{cases} \dot{S}[v] + [\sigma] = 0 \\ \dot{S}[\varepsilon] + [v] = 0 \\ \dot{S}(t)\left([\psi] - \frac{\sigma^+ + \sigma^-}{2}[\varepsilon]\right) \geq 0 \end{cases}$$

$$\begin{cases} (\varepsilon, \sigma, v)(X, t) = (\varepsilon, \sigma, v)(\xi) \\ \xi = X - \dot{S}t, \quad \dot{S} = const. \\ \lim_{\xi \to \pm\infty}(\varepsilon, \sigma, v)(\xi) = (\varepsilon^{\pm}, \sigma_{eq}(\varepsilon^{\pm}), v^{\pm}) \end{cases}$$

Uniqueness criterion: viscosity criterion $\Leftrightarrow$ chord criterion

Admissible solutions $\leftrightarrow$ Chord criterion.

$$\varrho\dot{S}^2 = \frac{\sigma^+ - \sigma^-}{\varepsilon^+ - \varepsilon^-} \geq (\leq) \frac{\sigma^+ - \sigma_{eq}(\varepsilon)}{\varepsilon^+ - \varepsilon} \quad \text{if} \quad \dot{S} > (<) 0, \quad \forall\, \varepsilon \in \varepsilon^+ \text{ and } \varepsilon^-$$

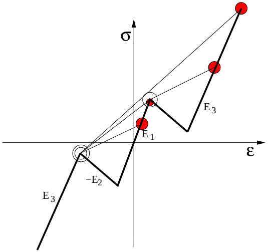


Fig. 1 Admissible waves connecting states: $(\varepsilon^+, \sigma^+)$ (empty circle) & $(\varepsilon^-, \sigma^-)$ (red circle), for $\varepsilon^+ < \varepsilon^-$ and $\dot{S} > 0$.

Traveling wave solutions $\leftrightarrow$ Shock layers

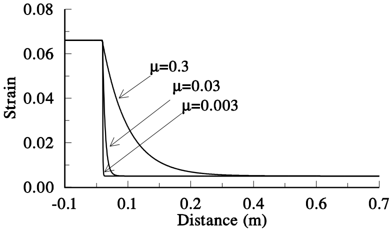
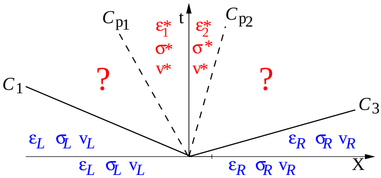


Fig. 2 Influence of the viscosity coefficient.

Riemann problem = Left Goursat problem + Right Goursat problem



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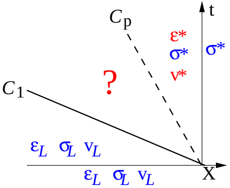

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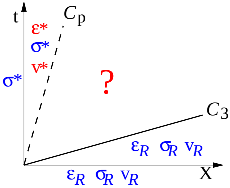
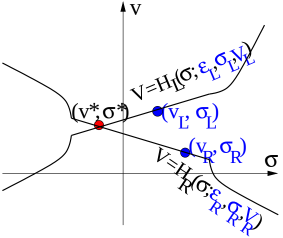


$$[\sigma] = 0, \quad [v] = 0 \quad \text{at } X = 0$$

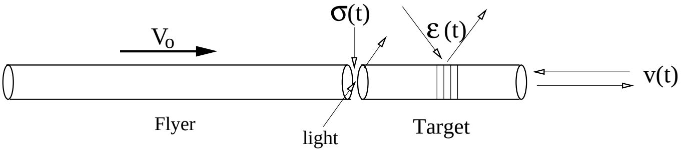
$$v^* = H_L(\sigma^*; \varepsilon_L, \sigma_L, v_L) \qquad v^* = H_R(\sigma^*; \varepsilon_R, \sigma_R, v_R)$$

Wave structure solution is obtained by intersecting the curves:



Proposed experiment:

longitudinal impact of two SMA bars



a methodology for investigating the dynamic material response

thermoelastic approach

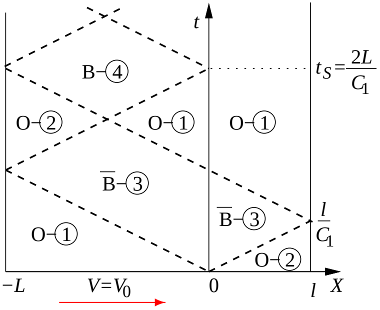
$$\sigma = \sigma_{eq}(\varepsilon, \theta)$$

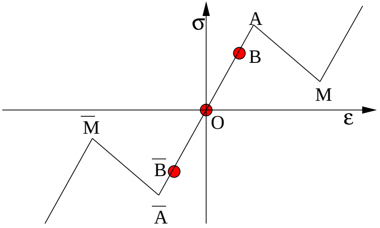
versus

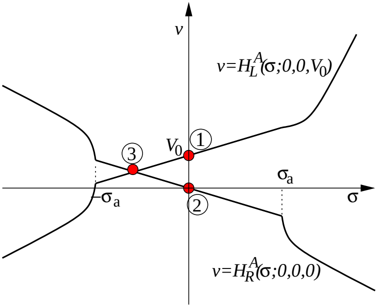
rate-type approach

$$\frac{\partial \sigma}{\partial t} - E\frac{\partial \varepsilon}{\partial t} = -\frac{E}{\mu}(\sigma - \sigma_{eq}(\varepsilon, \theta))$$

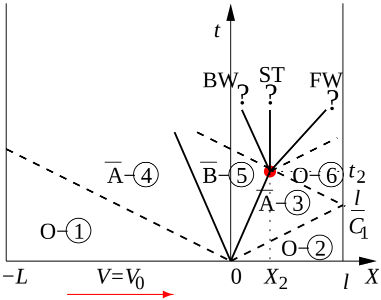
Elastic linear impact $\iff V_0 \leq V_{ph} \equiv \frac{2\sigma_a}{\sqrt{\varrho E_1}}$

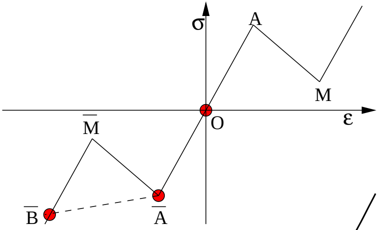


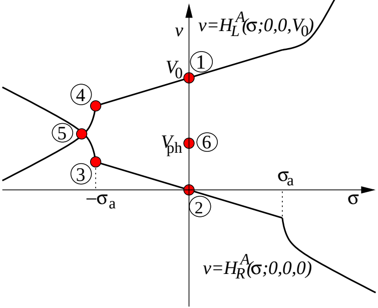




Impact induced phase transformation ($V_0 > V_{ph}$)



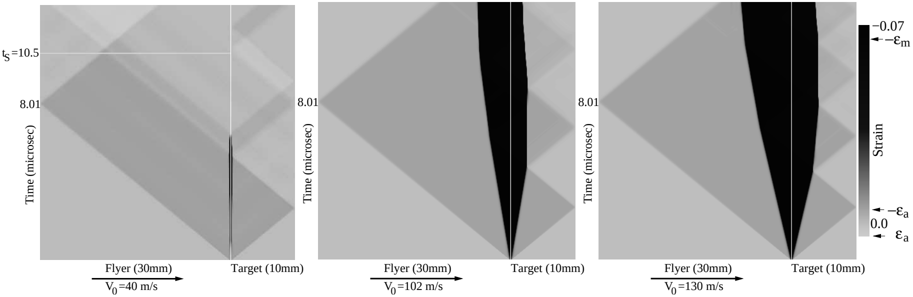




Influence of the impact velocity V_0

Rate-type model - numerical results

$$\mu = 10^{-3} MPa \times s \qquad a = \frac{E - E_1}{2E_1}\frac{\mu}{\varrho} = 3.8 \times 10^{-4} m^2/s$$

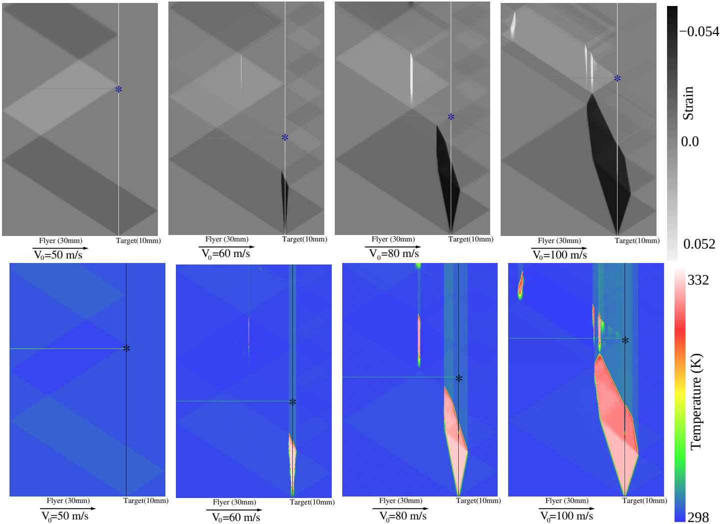


$V_0 = 40.m/s$ $\qquad$ $V_0 = 102.m/s$ $\qquad$ $V_0 = 130.m/s$

Critical velocities predicted by the elastic model:

$$V_{ph} = 37.2m/s \qquad V_{bw} = 75.4m/s \qquad V_{fw} = 102.5m/s$$

Influence of the impact velocity V_0 on the time of separation



Strain (phase) distribution and temperature distribution in the bars

Symbol * corresponds to the separation moment t_S.

Propagation of disturbances in a pure phase

$$\varrho\frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial X} = 0,$$

$$\frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial X} = 0,$$

$$\frac{\partial \sigma}{\partial t} - E\frac{\partial \varepsilon}{\partial t} = -\frac{E}{\mu}(\sigma - E_0\varepsilon + s),$$

$$E_0 = \begin{cases} E_1 > 0 & \text{in } \mathcal{A}\text{-phase} \\ E_3 > 0 & \text{in } \mathcal{M}^{\pm}\text{-phase} \\ -E_2 < 0 & \text{in } \textit{unstable}\text{-phases} \end{cases}$$

Behavior of the waves: $E > E_0$

- near the source - instantaneous waves $\rightarrow \sqrt{\frac{E}{\varrho}}$
- far from the source - delayed waves $\rightarrow \sqrt{\frac{E_0}{\varrho}}$

Initial value problem: $(\varepsilon, \sigma, v)(X, 0) = (\varepsilon^\star(X), \sigma^\star(X), v^\star(X)), \quad$ for $X \in R$

Method of multiple scales

Dimensionless system:

$T > 0 \to$ time scale; $\Omega T \to$ space scale

$$\begin{cases} \varrho \frac{\partial v}{\partial \tilde{t}} - \frac{\partial \sigma}{\partial \tilde{X}} = 0, \\ \frac{\partial \varepsilon}{\partial \tilde{t}} - \frac{\partial v}{\partial \tilde{X}} = 0, \\ \frac{\partial \sigma}{\partial \tilde{t}} - E\frac{\partial \varepsilon}{\partial \tilde{t}} = -E\eta(\sigma - E_0\varepsilon + s), \end{cases}$$

$\tilde{t} = \frac{t}{T}$ and $\tilde{X} = \frac{X}{\Omega T}$

$\eta = (\varrho\Omega^2)\frac{T}{\mu} \to$ perturbation parameter

Instantaneous wave solutions $\to$ small time scale T $\to$ small η

approximate solution:

$$v = v(\tilde{t}, \tilde{X}, \eta) = v_0(t_0, t_1, \ldots, X_0) + \eta v_1(t_0, t_1, \ldots, X_0) + \ldots$$

multiple variables: $X_0 = \tilde{X}$, $t_0 = \tilde{t}$, $t_1 = \eta\tilde{t}$, $\ldots$ $t_n = (\eta)^n\tilde{t}$, r

Delayed wave solutions $\to$ large time scale T $\to$ small $\frac{1}{\eta}$

$\vdots$

Instantaneous waves

Perturbations:

- **propagate near their sources with the speed** $\pm\sqrt{\frac{E}{\varrho}}$
- **are exponentially damped** $\rightarrow$ damping factor $= -\frac{E-E_0}{2\mu} < 0$
- **describe growth/decay of strain** $\rightarrow$ growth/damping factor $= -\frac{E_0}{\mu}$

$$v(t,X) \simeq \Big(\frac{v^*(\gamma)+v^*(\nu)}{2} + \frac{\sigma^*(\gamma)-\sigma^*(\nu)}{2\sqrt{\varrho E}}\Big)\exp\Big(-\frac{E-E_0}{2\mu}t\Big),$$

$$\sigma(t,X) \simeq \Big(\frac{\sigma^*(\gamma)+\sigma^*(\nu)}{2} - \frac{v^*(\gamma)-v^*(\nu)}{\sqrt{\varrho E}} + \frac{2E}{E-E_0}s\Big)\exp\Big(-\frac{E-E_0}{2\mu}t\Big) - \frac{2E}{E-E_0}s,$$

$$\varepsilon(t,X) \simeq \frac{1}{E}\sigma(t,X) + \Big(\varepsilon^*(X) - \frac{1}{E}\sigma^*(X) - \frac{1}{E_0}s\Big)\exp\Big(-\frac{E_0}{\mu}t\Big) + \frac{1}{E_0}s,$$

where $\gamma = X - \sqrt{\frac{E}{\varrho}}t$ and $\nu = X + \sqrt{\frac{E}{\varrho}}t$.

Delayed waves

Perturbations:

- **propagate far from their sources with the speed** $\pm\sqrt{\frac{E_0}{\varrho}}$
- **spread out like** $\sqrt{at}$
- **amplitude decreases like** $\frac{1}{\sqrt{at}}$

$$\left.\right\} a = \frac{E-E_0}{2E}\frac{\mu}{\varrho} \rightarrow \text{diffusion coefficient}$$

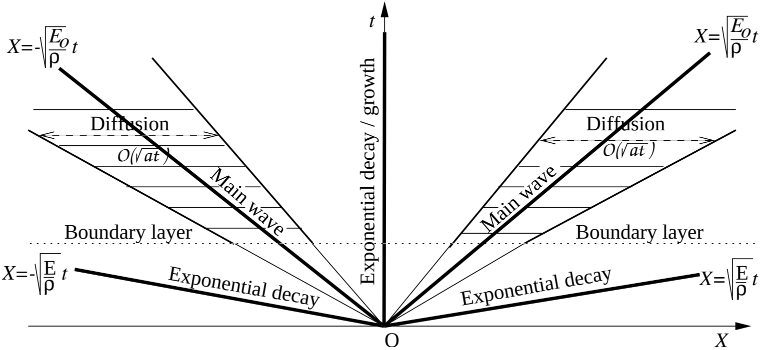
$$v(t,X) = \frac{1}{2\sqrt{\pi at}}\int_{-\infty}^{\infty}\left(G(s)\exp\left(-\frac{(s-X-\sqrt{E_0/\varrho}\ t)^2}{4at}\right)+F(s)\exp\left(-\frac{(s-X+\sqrt{E_0/\varrho}\ t)^2}{4at}\right)\right)ds$$

$$\varepsilon(t,X) = \frac{\sqrt{E_0/\varrho}}{2\sqrt{\pi at}}\int_{-\infty}^{\infty}\left(G(s)\exp\left(-\frac{(s-X-\sqrt{E_0/\varrho}\ t)^2}{4at}\right)-F(s)\exp\left(-\frac{(s-X+\sqrt{E_0/\varrho}\ t)^2}{4at}\right)\right)ds$$

$$\sigma(t,X) = C(X)\exp\left(-\frac{E}{\mu}t\right) + E_0\varepsilon(t,X) - s$$

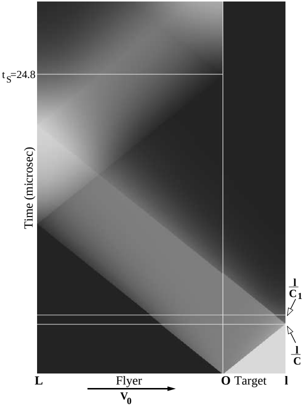
where $G(X)$, $F(X)$, $C(X) \leftrightarrow$ initial data

Schematic representation of the propagating fronts in a single phase for rate type materials

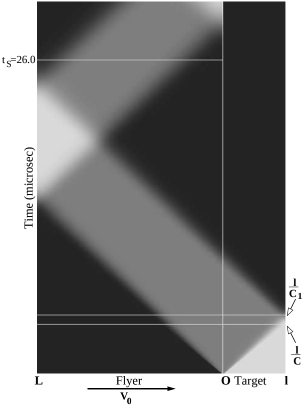


Impact in a pure phase: instantaneous and delayed waves

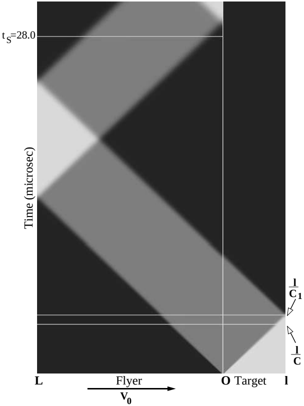
Diffusion coefficient: $a = \frac{E - E_1}{2E}\frac{\mu}{\varrho}$ Damping factor: $-\frac{E - E_1}{2\mu}$



$a = 0.8\ m^2/s$
(μ=0.04 $MPa \times s$)



$a = 0.08\ m^2/s$
(μ=0.004 $MPa \times s$)



$a = 0.008\ m^2/s$
(μ=0.0004 $MPa \times s$)

Conclusion

- Maxwellian rate-type model $\dot{\sigma} - E\dot{\varepsilon} = -\frac{E}{\mu}(\sigma - \sigma_{eq}(\varepsilon))$
  - describes both quasistatic and dynamic response of SMA
  - contains as a limit case ($E \to \infty$) Kelvin-Voigt model $\sigma = \sigma_{eq}(\varepsilon) + \mu\dot{\varepsilon}$
  - both viscous models lead to the same admissibility criterion

- Outlook
  - consider viscosity-capillarity model $\sigma = \sigma_{eq}(\varepsilon) + \mu\dot{\varepsilon} - \lambda\varepsilon_{xx}$
  - traveling wave analysis $\to$ admissible shocks violating Lax conditions.
  - different selection criteria furnish different unique solutions
  - only systematic experimental investigation could decide which is the physical relevant one