The Periodic Unfolding Method
and its Applications
in the Homogenization of Multi-scale Structures

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Survey on the

Periodic Unfolding Method

introduced with Alain Damlamian and Georges Griso, and on some of its applications to **homogenization problems for multi-scale structures and for non linear integral-type energies**.

Joint works with

Alain DAMLAMIAN, Université Paris 12 - Val de Marne

Riccardo DE ARCANGELIS , Università di Napoli

Georges Griso, Université Paris 6, Pierre et Marie Curie

Periodic Unfolding Method

Cioranescu D., Damlamian A., Griso G., *Periodic unfolding and homogenization*, C. R. Acad. Sci. Paris Sér. I Math. **335**, (2002), 99-104.

Cioranescu D., Damlamian A., Griso G., *The periodic unfolding method in homogenization*, To appear in SIAM J. Math. Analysis, 2008.

The Periodic unfolding Method is a "fixed domain" method which is well suited to approach different classes of periodic homogenization problems.

The basic idea is that, provided the proper scale is used, oscillatory behaviours can be turned into weak, or even strong, convergence, at the price of an increase of the space dimension of the problem, but with significant simplifications in the proofs.

In particular, in [Cioranescu, Damlamian, Griso] the homogenization of quadratic energies has been carried out, and it has been shown that the use of periodic unfolding method simplifies the homogenization process by actually reducing it to a weak convergence problem in L^2 spaces.

It also provides an elementary proof of the results of the theory of two-scale convergence of Nguetseng and Allaire.

The method was successfully applied to many problems (originally, to the homogenization of linear problems in the standard periodic case, then to the one with holes).
It also applies to

- elasticity and composite materials,
- truss-like structures, as well as rods, plates and composite thereof,
- multiscale periodic homogenization.

The aim here is to present some examples approached by means of the periodic unfolding method, in particular the homogenization of nonlinear convex or quasi-convex energies, as well as the homogenization of pointwise gradient constrained convex energies.

The Unfolding Operator and Its Main Properties

$\Omega \subseteq \mathbf{R}^n$ smooth bounded open set, Y reference cell (usually $Y =]0,1[^n$).

For $z \in \mathbf{R}^n$, $[z]$ is the vector whose coordinates are the integer parts of the corresponding ones of z.

For every $\varepsilon > 0$ the unfolding operator

$$\mathcal{T}_\varepsilon \colon L^1(\Omega) \to L^1(\mathbf{R}^n \times Y)$$

is defined by

$$\mathcal{T}_\varepsilon(v)(x,y) = \widetilde{v}\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon y\Big) \text{ for every } v \in L^1(\Omega), \text{ and a.e. } (x,y) \in \mathbf{R}^n \times Y,$$

where $\widetilde{v}$ is the extension of v by zero outside Ω.

$$\Omega_\varepsilon = \bigcup_{\xi \in \mathbf{Z}^n,\ \varepsilon(\xi + \overline{Y}) \cap \overline{\Omega} \neq \emptyset} \varepsilon(\xi + \overline{Y}).$$

Then one has

$$\int_{\Omega_\varepsilon \times Y} \mathcal{T}_\varepsilon(v)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega v(x)\,\mathrm{d}x \text{ for every } v \in L^1(\Omega), \text{ and } \varepsilon > 0.$$

Moreover,

$$\mathcal{T}_\varepsilon(v) \to \widetilde{v} \text{ in } L^p(\mathbf{R}^n \times Y) \text{ as } \varepsilon \to 0, \text{ for every } p \in [1,+\infty[, \text{ and } v \in L^p(\Omega).$$

If if $w = (w_1, \dots, w_n) \in L^1(\Omega)^n$, we set for every $\varepsilon > 0$

$$\mathcal{T}_\varepsilon(w) = (\mathcal{T}_\varepsilon(w_1), \dots, \mathcal{T}_\varepsilon(w_n)).$$

Proposition. *Let $p \in [1,+\infty[$, $\{\varepsilon_h\} \subseteq]0,+\infty[$ be strictly decreasing to 0. Let $\{v_h\}$ be a sequence converging weakly in $W^{1,p}(\Omega)$ to some v. Then, there exist a subsequence $\{h_k\}$, and some $V \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$ such that*

$$\mathcal{T}_{\varepsilon_{h_k}}(\nabla v_{h_k}) \to \nabla v + \nabla_y V \text{ weakly in } L^p(\Omega \times Y)^n.$$

Relation with the Two-Scale Convergence.

Proposition. *Let* $p \in]1, +\infty[$, $\{v_h\}$ *be a bounded sequence in* $L^p(\Omega)$, $v \in L^p(\Omega \times Y)$, *and* $\{\varepsilon_h\} \subseteq]0, +\infty[$ *be strictly decreasing to 0. Then* $\{\mathcal{T}_{\varepsilon_h}(v_h)\}$ *converges weakly to* v *in* $L^p(\Omega \times Y)$ *if and only if* $\{v_h\}$ *two-scale converges to* v.

$\{v_h\}$ two-scale converges to v if and only if

$$\lim_{h \to +\infty} \int_\Omega w_h(x) \varphi\Big(x, \frac{x}{\varepsilon_h}\Big) \, \mathrm{d}x = \int_{\Omega \times Y} v(x, y) \varphi(x, y) \, \mathrm{d}x \, \mathrm{d}y$$

for every $\{\varepsilon_h\} \subseteq]0, +\infty[$ strictly decreasing to 0 and every smooth φ on $\Omega \times Y$ Y-periodic in the second group of variables.

Homogenization of Nonlinear Convex Energies

Cioranescu D., Damlamian A., De Arcangelis R.: *Homogenization of Nonlinear Integrals via the Periodic Unfolding Method*; C. R. Acad. Sci. Paris Sér. I Math. **339**, (2004), 77-82.

Homogenization problem in the general case of nonlinear convex integral energies.

For such energies the homogenization result is well established, and goes back to [Marcellini P.: *Periodic Solutions and Homogenization of Non Linear Variational Problems*; Ann. Mat. Pura Appl. (4), **117**, (1978), 139-152] and [Carbone L., Sbordone C.: *Some Properties of* Γ*-Limits of Integral Functionals*; Ann. Mat. Pura Appl. (4), **122**, (1979), 1-60].

The use of periodic unfolding method simplifies the treatment of the homogenization process, and allows the deduction of different types of limit formulas, by reducing it again to a weak convergence problem in an appropriate L^p space.

$\mathcal{A}_0$ is the class of the bounded open subsets of $\mathbf{R}^n$ having Lipschitz boundary.

f Carathéodory energy density

$$(H_0) \qquad \begin{cases} f\colon (x,z) \in \mathbf{R}^n \times \mathbf{R}^n \mapsto f(x,z) \in [0,+\infty[\\ f(\cdot,z) \text{ Lebesgue measurable, and } Y\text{-periodic for every } z \in \mathbf{R}^n \\ f(x,\cdot) \text{ convex for a.e. } x \in \mathbf{R}^n. \end{cases}$$

For $p \in [1,+\infty[$, and $M > 0$

$$(H_1) \qquad f(x,z) \leq M(1+|z|^p) \text{ for a.e. } x \in \mathbf{R}^n, \text{ and every } z \in \mathbf{R}^n,$$

$$(H_2) \qquad |z|^p \leq f(x,z) \text{ for a.e. } x \in \mathbf{R}^n, \text{ and every } z \in \mathbf{R}^n.$$

$W^{1,p}_{\mathrm{per}}(Y)$ Banach space of Y-periodic functions in $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^n)$ endowed with the $W^{1,p}(Y)$-norm.

$$f_{\mathrm{hom}}\colon z \in \mathbf{R}^n \mapsto \inf\left\{ \int_Y f(y, z + \nabla v(y))\, \mathrm{d}y : v \in W^{1,p}_{\mathrm{per}}(Y) \right\}.$$

Theorem. *Let f satisfy (H_0), and assume that (H_1) holds for some $p \in [1,+\infty[$. Let $\Omega \in \mathcal{A}_0$, and let $\{\varepsilon_h\} \subseteq]0,+\infty[$ converge to 0. Then, for every $u \in W^{1,p}(\Omega)$,*

$$\inf\left\{\liminf_{h\to+\infty}\int_\Omega f\Big(\frac{x}{\varepsilon_h},\nabla u_h(x)\Big)\,\mathrm{d}x : \{u_h\}\subseteq W^{1,p}(\Omega),\ u_h \rightharpoonup u \text{ in } W^{1,p}(\Omega)\right\} =$$

$$= \inf\left\{\limsup_{h\to+\infty}\int_\Omega f\Big(\frac{x}{\varepsilon_h},\nabla u_h(x)\Big)\,\mathrm{d}x : \{u_h\}\subseteq W^{1,p}(\Omega),\ u_h \rightharpoonup u \text{ in } W^{1,p}(\Omega)\right\} =$$

$$= \inf\left\{\int_{\Omega\times Y} f(y,\nabla u(x)+\nabla_y V(x,y))\,\mathrm{d}x\,\mathrm{d}y : V \in L^p(\Omega;W^{1,p}_{\mathrm{per}}(Y))\right\}.$$

If in addition, (H_2) holds with $p \in]1,+\infty[$, then this common value equals

$$\int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x.$$

The proof is contained in the following three lemmas.

Lemma 1. *Assume that f satisfies (H_0). Let $\Omega \in \mathcal{A}_0$, $p \in [1,+\infty[$, and $\{\varepsilon_h\} \subseteq]0,+\infty[$ converge to 0 as $h \to +\infty$. Then, for every $u \in W^{1,p}(\Omega)$,*

$$\inf\left\{\liminf_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla u_h(x)\right) \mathrm{d}x : \{u_h\} \subseteq W^{1,p}(\Omega),\ u_h \rightharpoonup u \text{ in } W^{1,p}(\Omega)\right\} \geq$$

$$\geq \inf\left\{\int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y V(x,y))\,\mathrm{d}x\,\mathrm{d}y : V \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))\right\}.$$

Lemma 2. *Assume that f satisfies (H_0), and that (H_1) holds for some $p \in [1,+\infty[$. Let $\Omega \in \mathcal{A}_0$, and let $\{\varepsilon_h\} \subseteq]0,+\infty[$ converge to 0 as $h \to +\infty$. Then, for every $u \in W^{1,p}(\Omega)$,*

$$\inf\left\{\limsup_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla u_h(x)\right) \mathrm{d}x : \{u_h\} \subseteq W^{1,p}(\Omega),\ u_h \rightharpoonup u \text{ in } W^{1,p}(\Omega)\right\}$$

$$\leq \inf\left\{\int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y V(x,y))\,\mathrm{d}x\,\mathrm{d}y : V \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))\right\}.$$

Lemma 3. *Assume that f satisfies (H_0). Let $\Omega \in \mathcal{A}_0$, $p \in]1, +\infty[$, and suppose that (H_1) and (H_2) hold. Then, for every $u \in W^{1,p}(\Omega)$,*

$$\inf\left\{ \int_{\Omega \times Y} f(y, \nabla u(x) + \nabla_y V(x,y)) \, \mathrm{d}x \, \mathrm{d}y : V \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y)) \right\} = \int_\Omega f_{\mathrm{hom}}(\nabla u(x)) \, \mathrm{d}x.$$

The proof of each of these lemmas is elementary!!

Proof (of Lemma 1, easy). $u \in W^{1,p}(\Omega)$, $u_h \rightharpoonup u$, $\lim_{h\to+\infty} \int_\Omega f(\frac{x}{\varepsilon_h}, \nabla u_h(x))\,\mathrm{d}x$ exists and is finite. Then there exists $\{\varepsilon_{h_k}\} \subseteq \{\varepsilon_h\}$ and $U \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$ with

$$\mathcal{T}_{\varepsilon_{h_k}}(\nabla u_{h_k}) \rightharpoonup \nabla u + \nabla_y U \text{ in } (L^p(\Omega \times Y))^n.$$

$$\int_\Omega f\Big(\frac{x}{\varepsilon_{h_k}}, \nabla u_{h_k}(x)\Big)\,\mathrm{d}x = \int_{\Omega_{\varepsilon_{h_k}}\times Y} \mathcal{T}_{\varepsilon_{h_k}}\Big(f\Big(\frac{\cdot}{\varepsilon_{h_k}}, \nabla u_{h_k}(\cdot)\Big)\Big)(x,y)\,\mathrm{d}x\,\mathrm{d}y \geq$$

$$\geq \int_{\Omega\times Y} f\Big(\frac{1}{\varepsilon_{h_k}}\Big(\varepsilon_{h_k}\Big[\frac{x}{\varepsilon_{h_k}}\Big] + \varepsilon_{h_k} y\Big), \mathcal{T}_{\varepsilon_{h_k}}(\nabla u_{h_k})(x,y)\Big)\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \int_{\Omega\times Y} f\Big(\Big[\frac{x}{\varepsilon_{h_k}}\Big] + y, \mathcal{T}_{\varepsilon_{h_k}}(\nabla u_{h_k})(x,y)\Big)\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega\times Y} f(y, \mathcal{T}_{\varepsilon_{h_k}}(\nabla u_{h_k})(x,y))\,\mathrm{d}x\,\mathrm{d}y,$$

where the last functional is sequentially weakly $(L^p(\Omega))^n$-lower semicontinuous (well-known consequence of Fatou's Lemma under hypothesis (H_0)).

$$\liminf_{h\to+\infty} \int_\Omega f\Big(\frac{x}{\varepsilon_h}, \nabla u_h(x)\Big)\,\mathrm{d}x \geq \int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y U(x,y))\,\mathrm{d}x\,\mathrm{d}y \geq$$

$$\geq \inf\Big\{\int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y V(x,y))\,\mathrm{d}x\,\mathrm{d}y : V \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))\Big\}.$$

Proof (of Lemma 2). $u \in W^{1,p}(\Omega)$, $U \in C^1(\mathbf{R}^n \times \mathbf{R}^n)$ with $U(x,\cdot)$ Y-periodic for every $x \in \Omega$. For every $h \in \mathbf{N}$ and $x \in \Omega$, we set

$u_h(x) = u + \varepsilon_h U(x, \frac{x}{\varepsilon_h})$. Clearly,

$$\nabla u_h(x) = \nabla u(x) + \varepsilon_h \nabla_x U\Big(x, \frac{x}{\varepsilon_h}\Big) + \nabla_y U\Big(x, \frac{x}{\varepsilon_h}\Big) \text{ for every } h \in \mathbf{N},\ x \in \Omega.$$

Then, from the periodicity of U, one has

$$\int_\Omega f\Big(\frac{x}{\varepsilon_h}, \nabla u_h(x)\Big)\,\mathrm{d}x = \int_{\Omega_{\varepsilon_h}\times Y} f(y, \mathcal{T}_{\varepsilon_h}(\nabla u_h)(x,y))\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \int_{\Omega_{\varepsilon_h}\times Y} f\Big(y, \mathcal{T}_{\varepsilon_h}(\nabla u)(x,y) + \varepsilon_h \nabla_x U\Big(\varepsilon_h\Big[\frac{x}{\varepsilon_h}\Big] + \varepsilon_h y, y\Big) + \nabla_y U\Big(\varepsilon_h\Big[\frac{x}{\varepsilon_h}\Big] + \varepsilon_h y, y\Big)\Big)\,\mathrm{d}x\,\mathrm{d}y.$$

Due to continuity properties of $\nabla_x U$ and to the periodicity of $\nabla_y U$,

$$\varepsilon_h \nabla_x U\Big(\cdot, \frac{\cdot}{\varepsilon_h}\Big) \to 0 \text{ uniformly in } \Omega,$$

$$\nabla_y U\Big(\cdot, \frac{\cdot}{\varepsilon_h}\Big) \to \int_Y \nabla_y U(\cdot, y)\,\mathrm{d}y = 0 \text{ weakly* in } (L^\infty(\Omega))^n.$$

This implies that $u_h \rightharpoonup u$ in $W^{1,p}(\Omega)$. Moreover,

$$\inf\left\{\limsup_{h\to+\infty}\int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla v_h(x)\right)\,\mathrm{d}x : \{v_h\}\subseteq W^{1,p}(\Omega),\ v_h \rightharpoonup u \text{ in } W^{1,p}(\Omega)\right\}$$

$$\leq \limsup_{h\to+\infty}\int_{\Omega_{\varepsilon_h}\times Y} f\left(y, \mathcal{T}_{\varepsilon_h}(\nabla u)(x,y) + \varepsilon_h \nabla_x U\left(\varepsilon_h\left[\frac{x}{\varepsilon_h}\right] + \varepsilon_h y, y\right)\right.$$

$$\left. + \nabla_y U\left(\varepsilon_h\left[\frac{x}{\varepsilon_h}\right] + \varepsilon_h y, y\right)\right)\,\mathrm{d}x\,\mathrm{d}y.$$

On the other hand, again by the continuity properties of $\nabla_x U$ and of $\nabla_y U$,

$$\varepsilon_h \nabla_x U\left(\varepsilon_h\left[\frac{\cdot}{\varepsilon_h}\right] + \varepsilon_h\cdot, \cdot\right) \to 0 \text{ and}$$
$$\nabla_y U\left(\varepsilon_h\left[\frac{\cdot}{\varepsilon_h}\right] + \varepsilon_h\cdot, \cdot\right) \to \nabla_y U \text{ uniformly in } \Omega.$$

One also has $\mathcal{T}_{\varepsilon_h}(\nabla u) \to \nabla u$ in $(L^p(\mathbf{R}^n \times Y))^n$. So we can pass to the limit and using (H_1), we get

$$\lim_{h\to+\infty}\int_{\Omega_{\varepsilon_h}\times Y} f\Big(y, \mathcal{T}_{\varepsilon_h}(\nabla u)(x,y) + \varepsilon_h \nabla_x U\Big(\varepsilon_h\Big[\frac{x}{\varepsilon_h}\Big] + \varepsilon_h y, y\Big) +$$

$$+\nabla_y U\Big(\varepsilon_h\Big[\frac{x}{\varepsilon_h}\Big] + \varepsilon_h y, y\Big)\Big)\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y U(x,y))\,\mathrm{d}x\,\mathrm{d}y.$$

We conclude the proof by a standard density argument when $U \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$ (by observing that (H_0) and (H_1) imply the continuity on $L^p(\Omega; W^{1,p}_{per}(Y))$ of the right hand side in the above inequality).

Proof (of Lemma 3).

After the two steps what we know is that the "limit" of the functional $\int_\Omega f(\frac{x}{\varepsilon}, \nabla u_\varepsilon)\ \mathrm{d}x$ is

$$\inf\left\{\int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y V(x,y))\ \mathrm{d}x\ \mathrm{d}y : V \in L^p(\Omega; W^{1,p}_{per}(Y))\right\}.$$

We want to prove that this value is nothing else than $\int_\Omega f^p_{\mathrm{hom}}(\nabla u(x))\ \mathrm{d}x$. From the definition of f^p_{hom}, we have

$$\begin{aligned}&\int_\Omega f^p_{\mathrm{hom}}(\nabla u(x))\ \mathrm{d}x\\ &\quad = \inf\left\{\int_Y f(y, \nabla u(x) + \nabla v(y))\ \mathrm{d}y : v \in W^{1,p}_{per}(Y)\right\}.\end{aligned}$$

One inequality is straightforward, since for u in $W^{1,p}(\Omega)$ and V in $L^p(\Omega; W^{1,p}_{per}(Y))$, the following inequality holds for a.e. $x \in \Omega$,

$$\int_Y f(y, \nabla u(x) + \nabla_y V(x,y))\ \mathrm{d}y \geq f^p_{\mathrm{hom}}(\nabla u(x)).$$

The reverse inequality is obvious if $\int_\Omega f^p_{\rm hom}(\nabla u(x))\,{\rm d}x = +\infty$. To prove it in the case where $\int_\Omega f^p_{\rm hom}(\nabla u(x))\,{\rm d}x < +\infty$, we make use of Castaing's selection theorem.

Castaing's theorem on measurable selections.

Let $\mathcal{O}$, X be sets, and $\mathcal{G}$ a multifunction from $\mathcal{O}$ to X. A function $\sigma\colon\Omega \to X$ will be said to be a selection of $\mathcal{G}$ if $\sigma(x) \in \mathcal{G}(x)$ for every $x \in \mathcal{O}$. The measurable selection result below is is known as Castaing's theorem.

Theorem. Let X be a separable metric space, $(\mathcal{O}, \mathcal{M})$ a measurable space, and $\mathcal{G}$ a multifunction from $\mathcal{O}$ to X. Assume that for every $x \in \mathcal{O}$, $\mathcal{G}(x)$ is nonempty and complete in X. Assume moreover, that for every closed subset F of X, $\{x \in \mathcal{O} : \mathcal{G}(x) \cap F \neq \emptyset\}$ belongs to $\mathcal{M}$. Then $\mathcal{G}$ admits a $\mathcal{M}$-measurable selection..

Note first, that due to (H_0) and (H_1), f^p_{hom} is convex and continuous on $\mathbf{R}^n$. Due to (H_2) and the Poincaré-Wirtinger Inequality, the infimum defining $f^p_{\text{hom}}(z)$, is achieved for every $z \in \mathbf{R}^n$.

This, and (H_1) imply that for $z \in \mathbf{R}^n$, $\Gamma(z)$ is nonempty, and strongly closed, where Γ is the multifunction defined by

$$\Gamma(z) \doteq \Big\{ v \in W^{1,p}_{per}(Y) : \int_Y v(y)\,\mathrm{d}y = 0, \\ \int_Y f(y, z + \nabla v(y))\,\mathrm{d}y = f^p_{\text{hom}}(z) \Big\}.$$

Now, by Castaing's Theorem, Γ has a $\mathcal{B}(\mathbf{R}^n)$-measurable selection, where $\mathcal{B}(\mathbf{R}^n)$ denotes the Borel σ-algebra of $\mathbf{R}^n$.

Let σ denote such a measurable selection. Fix $u \in W^{1,p}(\Omega)$. For a.e. $x \in \Omega$, set $U(x) = \sigma(\nabla u(x))$. Then U is $\mathcal{L}(\Omega)$-measurable, with values in $W^{1,p}_{per}(Y)$ so that

$$f^p_{\text{hom}}(\nabla u(x)) = \int_Y f(y, \nabla u(x) + \nabla_y U(x)(y))\,\mathrm{d}y \text{ for a.e. } x \in \Omega.$$

Integrating over Ω yields

$$\int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y U(x)(y))\,\mathrm{d}y\,\mathrm{d}x < +\infty,$$

so that by (H_2), $\nabla_y U$ is in $(L^p(\Omega\times Y))^n$. By the Poincaré-Wirtinger inequality, U belongs to $L^p(\Omega; W^{1,p}_{per}(Y))$, hence the claim

$$\int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x = \int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y U(x)(y))\,\mathrm{d}x\,\mathrm{d}y$$

$$\geq \inf\left\{\int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y V(x,y))\,\mathrm{d}x\,\mathrm{d}y : V \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))\right\}.$$

Ingredients

Lemma 1: Periodic Unfolding, Lower Semicontinuity (convexity).

Lemma 2: Periodic Unfolding, Growth Conditions.

Lemma 3: Castaing's Selection Theorem, Lower Semicontinuity (convexity), Coerciveness.

Homogenization of Quasiconvex Energies

CIORANESCU D., DAMLAMIAN A., DE ARCANGELIS R.: *Homogenization of Quasiconvex Integrals via the Periodic Unfolding Method*, SIAM Journal of Math. Anal., Vol. 37, 5 (2006), 1435-1453.

Homogenization problem in the general case of quasiconvex integral energies, under p-growth and coerciveness assumptions, defined on vector-valued configurations.

For such energies the homogenization result is well established, and goes back to [BRAIDES A.: *Homogenization of Some Almost Periodic Coercive Functional*; Rend. Accad. Naz. Sci. XL Mem. Mat. **9**, (1985), 313-322] and [MÜLLER S.: *Homogenization of Nonconvex Integral Functionals and Cellular Elastic Materials*; Arch. Rational Mech. Anal. **99**, (1987), 189-212].

We show how the use of periodic unfolding simplifies the treatment by reducing it again to a weak convergence problem.

$m,\ n \in \mathbf{N}.$

f Carathéodory energy density

$$(H_0) \qquad \begin{cases} f\colon (x,z) \in \mathbf{R}^n \times \mathbf{R}^{nm} \mapsto f(x,z) \in [0,+\infty[, \\ f(\cdot,z) \text{ Lebesgue measurable and } Y\text{-periodic for every } z \in \mathbf{R}^{nm}, \\ f(x,\cdot) \text{ continuous for a.e. } x \in \mathbf{R}^n. \end{cases}$$

$$(QC) \qquad f(x,\cdot) \text{ is quasiconvex for a.e. } x \in \mathbf{R}^n.$$

$p \in [1,+\infty[,\ M > 0,\ a \in L^1(Y)$ Y-periodic

$$(H_1) \qquad f(x,z) \le a(x) + M|z|^p \quad \text{for a.e. } x \in \mathbf{R}^n, \text{ and every } z \in \mathbf{R}^{nm},$$

$$(H_2) \qquad |z|^p \le f(x,z) \quad \text{for a.e. } x \in \mathbf{R}^n, \text{ and every } z \in \mathbf{R}^{nm}.$$

$$f_{\text{hom}}\colon z \in \mathbf{R}^{nm} \mapsto \lim_{t\to+\infty} \frac{1}{t^n} \inf\left\{ \int_{tY} f(y, z+\nabla v)\,\mathrm{d}y : v \in W_0^{1,p}(tY;\mathbf{R}^m) \right\}.$$

Theorem. *Let f satisfy (H_0) and (QC). Let $p \in \,]1,+\infty[$, and assume that (H_1) and (H_2) hold. Then, for every $\{\varepsilon_h\} \subseteq \,]0,+\infty[$ converging to 0, $\Omega \in \mathcal{A}_0$, and u in $W^{1,p}(\Omega;\mathbf{R}^m)$,*

$$\inf\left\{\liminf_{h\to+\infty}\int_\Omega f\Big(\frac{x}{\varepsilon_h},\nabla u_h\Big)\,\mathrm{d}x : \{u_h\}\subseteq W^{1,p}(\Omega;\mathbf{R}^m),\ u_h\to u \text{ in } L^p(\Omega;\mathbf{R}^m)\right\} =$$

$$= \inf\left\{\limsup_{h\to+\infty}\int_\Omega f\Big(\frac{x}{\varepsilon_h},\nabla u_h\Big)\,\mathrm{d}x : \{u_h\}\subseteq W^{1,p}(\Omega;\mathbf{R}^m),\ u_h\to u \text{ in } L^p(\Omega;\mathbf{R}^m)\right\} =$$

$$= \inf\left\{\liminf_{h\to+\infty}\int_\Omega f\Big(\frac{x}{\varepsilon_h},\nabla u_h\Big)\,\mathrm{d}x : \{u_h\}\subseteq u + W_0^{1,p}(\Omega;\mathbf{R}^m),\ u_h\to u \text{ in } L^p(\Omega;\mathbf{R}^m)\right\} =$$

$$= \inf\left\{\limsup_{h\to+\infty}\int_\Omega f\Big(\frac{x}{\varepsilon_h},\nabla u_h\Big)\,\mathrm{d}x : \{u_h\}\subseteq u + W_0^{1,p}(\Omega;\mathbf{R}^m),\ u_h\to u \text{ in } L^p(\Omega;\mathbf{R}^m)\right\} =$$

$$= \int_\Omega f_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x.$$

In the vector-valued case, for $u \in W^{1,p}(\Omega)$, the quantity

$$\lim_{t\to+\infty} \frac{1}{t^n} \inf\left\{\int_{\Omega\times tY} f(y, \nabla u(x) + \nabla_y V(x,y))\, \mathrm{d}x\, \mathrm{d}y : V \in L^p(\Omega; W_0^{1,p}(tY; \mathbf{R}^m))\right\}$$

plays the role of

$$\inf\left\{\int_{\Omega\times Y} f(y, \nabla u(x) + \nabla_y V(x,y))\, \mathrm{d}x\, \mathrm{d}y : V \in L^p(\Omega; W_{\mathrm{per}}^{1,p}(Y))\right\}.$$

Therefore,

Proposition. *Assume that f satisfies (H_0), (QC), (H_1), and (H_2) for some $p \in]1, +\infty[$. Then, for every $\Omega \in \mathcal{A}_0$ and $u \in W^{1,p}(\Omega; \mathbf{R}^m)$, the limit below exists, and*

$$\lim_{t\to+\infty} \frac{1}{t^n} \inf\left\{\int_{\Omega\times tY} f(y, \nabla u(x) + \nabla_y V(x,y))\, \mathrm{d}x\, \mathrm{d}y : V \in L^p(\Omega; W_0^{1,p}(tY; \mathbf{R}^m))\right\} =$$

$$= \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\, \mathrm{d}x.$$

Proof. Passage to the limit on t after an application of Castaing's Selection Theorem.

Various formulation for the limit energy:

$$\lim_{t\to+\infty}\frac{1}{t^n}\inf\left\{\int_{\Omega\times tY} f(y,\nabla u(x)+\nabla_y V(x,y))\,\mathrm{d}x\,\mathrm{d}y : V\in L^p(\Omega;W_0^{1,p}(tY;\mathbf{R}^m))\right\}=$$

$$=\lim_{t\to+\infty}\inf\left\{\int_{\Omega\times Y} f(ty,\nabla u(x)+\nabla_y V(x,y))\,\mathrm{d}x\,\mathrm{d}y : V\in L^p(\Omega;W_0^{1,p}(Y;\mathbf{R}^m))\right\}=$$

$$=\inf_{h\in\mathbf{N}}\inf\left\{\int_{\Omega\times Y} f(hy,\nabla u(x)+\nabla_y V(x,y))\,\mathrm{d}x\,\mathrm{d}y : V\in L^p(\Omega;W_0^{1,p}(Y;\mathbf{R}^m))\right\}.$$

The proof of the homogenization theorem reduces to the following lemmas.

The presence of an additional parameter complicates the computations.

Lemma 1. *Assume that f satisfies (H_0), (QC), (H_1), and (H_2) for some $p \in [1, +\infty[$. Let $\{\varepsilon_h\} \subseteq]0, +\infty[$ converge to 0, Ω in $\mathcal{A}_0$, and u in $W^{1,p}(\Omega; \mathbf{R}^m)$. Then*

$$\inf\left\{ \liminf_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla u + \nabla u_h\right) \mathrm{d}x : \{u_h\} \subseteq W_0^{1,p}(\Omega; \mathbf{R}^m),\ u_h \to 0 \text{ in } L^p(\Omega; \mathbf{R}^m) \right\} \geq$$

$$\geq \liminf_{h\to+\infty} \inf\left\{ \int_{\Omega\times Y} f(hy, \nabla u(x) + \nabla_y V(x,y))\, \mathrm{d}x\, \mathrm{d}y : V \in L^p(\Omega; W_0^{1,p}(Y; \mathbf{R}^m)) \right\}.$$

Lemma 2. *Assume that f satisfies (H_0) and (H_1) for some $p \in]1, +\infty[$. Let $\{\varepsilon_h\} \subseteq]0, +\infty[$ converge to 0, Ω in $\mathcal{A}_0$, and u in $W^{1,p}(\Omega; \mathbf{R}^m)$. Then*

$$\inf\left\{ \limsup_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla u_h\right) \mathrm{d}x : \{u_h\} \subseteq W^{1,p}(\Omega; \mathbf{R}^m),\ u_h \to u \text{ in } L^p(\Omega; \mathbf{R}^m) \right\} \leq$$

$$\leq \inf_{k\in\mathbf{N}} \frac{1}{k^n} \inf\left\{ \int_{\Omega\times kY} f(y, \nabla u(x) + \nabla_y V(x,y))\, \mathrm{d}x\, \mathrm{d}y : V \in L^p(\Omega; W_{\mathrm{per}}^{1,p}(kY; \mathbf{R}^m)) \right\}.$$

Proof (of Lemma 1). $u \in W^{1,p}(\Omega;\mathbf{R}^m)$.

Parameter doubling to keep into account the asymptotic homogenization formula:

$$\inf\left\{\liminf_{h\to+\infty}\int_\Omega f\left(\frac{x}{\varepsilon_h},\nabla u+\nabla u_h\right)\mathrm{d}x : \{u_h\}\subseteq W_0^{1,p}(\Omega;\mathbf{R}^m),\ u_h\to 0 \text{ in } L^p(\Omega;\mathbf{R}^m)\right\}\geq$$

$$\geq \sup_{\nu\in\mathbf{N}}\inf\left\{\liminf_{h\to+\infty}\int_\Omega f(\nu h x,\nabla u+\nabla v_h)\,\mathrm{d}x : \{v_h\}\subseteq W_0^{1,p}(\Omega;\mathbf{R}^m),\ v_h\to 0 \text{ in } L^p(\Omega;\mathbf{R}^m)\right\}.$$

Then, Periodic Unfolding Method coupled with the **De Giorgi's localization argument** because of the zero boundary datum constraint in the homogenization formula.

$$\sup_{\nu\in\mathbf{N}}\inf\left\{\liminf_{h\to+\infty}\int_\Omega f(\nu h x,\nabla u+\nabla v_h)\,\mathrm{d}x : \{v_h\}\subseteq W_0^{1,p}(\Omega;\mathbf{R}^m),\ v_h\to 0 \text{ in } L^p(\Omega;\mathbf{R}^m)\right\}\geq$$

$$\geq \liminf_{\nu\to+\infty}\lim_{h\to+\infty}\inf\left\{\int_{\Omega\times Y} f(hy,\mathcal{T}_{1/\nu}(\nabla u(x))+\nabla_y V(x,y))\,\mathrm{d}x\,\mathrm{d}y : V\in L^p(\Omega;W_0^{1,p}(Y;\mathbf{R}^m))\right\}.$$

Finally, the quasiconvexity of f and the properties of the unfolding operator provide

$$\liminf_{\nu\to+\infty} \lim_{h\to+\infty} \inf\left\{\int_{\Omega\times Y} f(hy, \mathcal{T}_{1/\nu}(\nabla u(x)) + \nabla_y V(x,y)) \, \mathrm{d}x \, \mathrm{d}y : \right.$$

$$\left. V \in L^p(\Omega; W_0^{1,p}(Y;\mathbf{R}^m))\right\} \geq$$

$$\geq \liminf_{h\to+\infty} \inf\left\{\int_{\Omega\times Y} f(hy, \nabla u(x) + \nabla_y V(x,y)) \, \mathrm{d}x \, \mathrm{d}y : V \in L^p(\Omega; W_0^{1,p}(Y;\mathbf{R}^m))\right\}.$$

Proof (of Lemma 2). $k \in \mathbf{N}$, $U \in C^1(\mathbf{R}^n \times \mathbf{R}^n; \mathbf{R}^m)$ with $U(x,\cdot)$ kY-periodic for every $x \in \Omega$.

$u_h(x) = u(x) + \varepsilon_h U(x, \frac{x}{\varepsilon_h})$.

Parameter doubling.

$$\int_\Omega f\Big(\frac{x}{\varepsilon_h}, \nabla u_h\Big) \, \mathrm{d}x = \int_{\Omega_{k\varepsilon_h}\times Y} \mathcal{T}_{k\varepsilon_h}\Big(f\Big(\frac{\cdot}{\varepsilon_h}, \nabla u_h(\cdot)\Big)\Big)(x,y) \, \mathrm{d}x \, \mathrm{d}y =$$

$$= \int_{\Omega_{k\varepsilon_h}\times Y} f\Big(ky, \mathcal{T}_{k\varepsilon_h}(\nabla u)(x,y) + \varepsilon_h \nabla_x U\Big(k\varepsilon_h\Big[\frac{x}{k\varepsilon_h}\Big] + k\varepsilon_h y, ky\Big) +$$

$$+ \nabla_y U\Big(k\varepsilon_h\Big[\frac{x}{k\varepsilon_h}\Big] + k\varepsilon_h y, ky\Big)\Big)\,\mathrm{d}x\,\mathrm{d}y.$$

As h diverges,

$$\inf\Big\{\limsup_{h\to+\infty}\int_\Omega f\Big(\frac{x}{\varepsilon_h}, \nabla u_h\Big)\,\mathrm{d}x : \{v_h\} \subseteq W^{1,p}(\Omega;\mathbf{R}^m),\ v_h \to u \text{ in } L^p(\Omega;\mathbf{R}^m)\Big\} \le$$

$$\le \frac{1}{k^n}\int_{\Omega\times kY} f(y, \nabla u(x) + \nabla_y U(x,y))\,\mathrm{d}x\,\mathrm{d}y,$$

for every $U \in C^1(\mathbf{R}^n \times \mathbf{R}^n;\mathbf{R}^m)$ with $U(x,\cdot)$ kY-periodic for every $x \in \Omega$.

A density argument, and the consideration of the infimum on k, completes the proof.

Homogenization of Pointwise Gradient Constrained Convex Energies

Cioranescu D., Damlamian A., De Arcangelis R.: *Homogenization of Integrals with Pointwise Gradient Constraints via the Periodic Unfolding Method*, Ricerche di Matematica, Vol. 55, 1 (2007), 31-54.

Homogenization problem in presence of pointwise oscillating gradient constraints.

Conjecture in [Bensoussan A., Lions J.L., Papanicolaou G.: "Asymptotic Analysis for Periodic Structures"; Stud. Math. Appl. **5**, North Holland (1978)], [Carbone L., Salerno S.: *Further Results on a Problem of Homogenization with Constraints on the Gradient*; J. Analyse Math. **44**, (1984/85), 1-20], [Corbo Esposito A., De Arcangelis R.: *Homogenization of Dirichlet Problems with Nonnegative Bounded Constraints on the Gradient*; J. Analyse Math. **64**, (1994), 53-96], [Carbone L., De Arcangelis R.: "Unbounded Functionals in the Calculus of Variations. Representation, Relaxation, and Homogenization"; Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math. **125**, Chapman & Hall/CRC, Boca Raton, FL (2001)].

Measure theoretic arguments.

The Periodic Unfolding Method provides a simpler and more direct proof.

f energy density

$$(H_0) \qquad \begin{cases} f\colon (x,z) \in \mathbf{R}^n \times \mathbf{R}^n \mapsto f(x,z) \in [0,+\infty] \\ f \ \mathcal{L}(\mathbf{R}^n) \times \mathcal{B}(\mathbf{R}^n)\text{-measurable} \\ f(\cdot,z) \ Y\text{-periodic for every } z \in \mathbf{R}^n, \ f(x,\cdot) \text{ convex for a.e. } x \in \mathbf{R}^n. \end{cases}$$

$p \in [1,+\infty]$, $a \in L^1_{\text{per}}(Y)$, $M \geq 0$, $r > 0$

$$(H_1) \qquad f(x,z) \leq a(x) + M|z|^p \text{ for a.e. } x \in \mathbf{R}^n \text{ and every } z \in \text{dom} f(x,\cdot),$$

$$(H_2) \qquad |z|^p \leq f(x,z) \text{ for a.e. } x \in \mathbf{R}^n \text{ and every } z \in \mathbf{R}^n,$$

$$(B) \qquad B_r \subseteq \text{dom} f(x,\cdot) \text{ for a.e. } x \in \mathbf{R}^n.$$

$$f_{\text{hom}}\colon z \in \mathbf{R}^n \mapsto \inf\left\{\int_Y f(y, z + \nabla v(y))\,\mathrm{d}y : v \in W^{1,p}_{\text{per}}(Y)\right\}.$$

Theorem. *Let* $p \in \,]n, +\infty[$, $a \in L^1_{\text{per}}(Y)$, $M \geq 0$, $r > 0$, *let* f *satisfy* (H_0), (H_1), (H_2), *(B). Then* f_{hom} *is convex, lower semicontinuous, and*

$$|z|^p \leq f_{\text{hom}}(z) \text{ for every } z \in \mathbf{R}^n,$$

$$B_r \subseteq \mathrm{dom} f_{\text{hom}}.$$

Let Ω *be a convex bounded open set, and let* $\{\varepsilon_h\} \subseteq \,]0, +\infty[$ *converge to 0. Then, for every* u *in* $W^{1,p}(\Omega)$,

$$\inf\left\{\liminf_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla u_h(x)\right) \mathrm{d}x : \{u_h\} \subseteq W^{1,p}(\Omega),\ u_h \to u \text{ in } C^0(\overline{\Omega})\right\} =$$

$$= \inf\left\{\limsup_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, \nabla u_h(x)\right) \mathrm{d}x : \{u_h\} \subseteq W^{1,p}(\Omega),\ u_h \to u \text{ in } C^0(\overline{\Omega})\right\} =$$

$$= \inf\left\{\int_{\Omega\times Y} f(y,\nabla u(x){+}\nabla_y V(x,y))\,\mathrm{d}x\,\mathrm{d}y : V\in L^p(\Omega;W^{1,p}_{\mathrm{per}}(Y))\right\} = \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x.$$

As consequence, for every $u\in W^{1,p}_0(\Omega)$,

$$\inf\left\{\liminf_{h\to+\infty}\int_\Omega f\left(\frac{x}{\varepsilon_h},\nabla u_h\right)\mathrm{d}x : \{u_h\}\subseteq W^{1,p}_0(\Omega),\ u_h\to u \text{ in } C^0(\overline{\Omega})\right\} =$$

$$= \inf\left\{\limsup_{h\to+\infty}\int_\Omega f\left(\frac{x}{\varepsilon_h},\nabla u_h\right)\mathrm{d}x : \{u_h\}\subseteq W^{1,p}_0(\Omega),\ u_h\to u \text{ in } C^0(\overline{\Omega})\right\} =$$

$$= \int_\Omega f_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x.$$

As corollaries, homogenization results under the "model case" assumptions

$$f(x,z) \le a(x) \text{ for a.e. } x \in \mathbf{R}^n, \text{ and every } z \in \operatorname{dom} f(x,\cdot),$$

$$\operatorname{dom} f(x,\cdot) \subseteq \overline{B_R} \text{ for a.e. } x \in \mathbf{R}^n,$$

where $a \in L^1_{\text{per}}(Y)$, and $R > 0$.

Since f may take the value $+\infty$, the above integrals involve pointwise gradient constrains. Indeed, for fixed $h \in \mathbf{N}$, the configurations v that make the integral $\int_\Omega f(\frac{x}{\varepsilon_h}, \nabla v)\,\mathrm{d}x$ finite must satisfy the constraint $\nabla v(x) \in \operatorname{dom} f(\frac{x}{\varepsilon_h}, \cdot)$ for a.e. $x \in \Omega$. This occurs in the "classical" elasto-plastic torsion case when $f(x,z) = |z|^2 + I_{[0,\varphi(\frac{x}{\varepsilon_h})]}(|z|)$, where φ is a nonnegative Y-periodic function in $L^\infty(Y)$, and $I_{[0,\varphi(\frac{x}{\varepsilon_h})]}(|z|)$ is the indicator function of $[0, \varphi(\frac{x}{\varepsilon_h})]$. This corresponds to the gradient constraint $|\nabla v(x)| \le \varphi(\frac{x}{\varepsilon_h})$ for a.e. $x \in \Omega$.

Ingredients: Periodic unfolding method, Abstract inner regularity results (hold only for convex open sets).