INTERNAL NONNEGATIVE STABILIZATION FOR SOME PARABOLIC EQUATIONS

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Abstract. We investigate the internal zero stabilization for some parabolic equations with state constraints. We also consider a two-component Reaction-Diffusion system posed on non coincident spatial domains and featuring a reaction term involving an integral kernel. The question of global existence of componentwise nonnegative solutions is assessed. Then we investigate the stabilization to zero of one of the solution components via an internal control distributed on a small subdomain while preserving nonnegativity of both components. Our results apply to predator-prey systems.

1. **A parabolic equation.** We shall investigate the internal zero stabilization for a parabolic equation with state constraints. Let $\Omega \subset \mathbf{R}^N$, $N \geq 2$, be a bounded domain and $\omega \subset\subset \Omega$ be a nonempty open subset, both with smooth enough boundaries $\partial\Omega$ and $\partial\omega$, respectively.

We shall study for the beginning the following problem:

$$\begin{cases} y_t(x,t) - div_x(a(x)\nabla_x y(x,t)) \\ \qquad +c(x)y(x,t) = m(x)u(x,t), & x \in \Omega,\ t > 0, \\ \partial_\nu y(x,t) + \alpha(x)y(x,t) = 0, & x \in \partial\Omega,\ t > 0 \\ y(x,0) = y_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where

m is the characteristic function of ω;
$a(x) = (a^{ij}(x))_{i,j=\overline{1,N}}$ is a quadratic selfadjoint matrix;
$a^{ij} \in C^1(\overline{\Omega})$, $i, j \in \{1, 2, ..., N\}$ and
there exists $a_0 > 0$ such that

$$(a(x)\zeta) \cdot \zeta = a^{ij}(x)\zeta_i\zeta_j \geq a_0|\zeta|^2, \quad \forall \zeta = (\zeta_1, \zeta_2, ..., \zeta_N) \in \mathbf{R}^N;$$

$\alpha \in L^\infty(\partial\Omega)$, $\alpha(x) \geq 0$ a.e. $x \in \partial\Omega$;
$c \in L^\infty(\Omega)$.

The summation convention that repeated indices indicate summation from 1 to N is followed here as it will be throughout.

Here, $\partial_\nu z$ is the conormal derivative on $\partial\Omega$, i.e.,

$$\partial_\nu z(x) = (a(x)\nabla z(x)) \cdot \nu(x) = a^{ij}(x)\nu_i(x)D_j z(x),$$

where $\nu(x) = (\nu_1(x), \nu_2(x), ..., \nu_N(x))$ is the outward normal versor to $\partial\Omega$ at x and u is the control and acts on ω (u is an internal control).

Remark 1. If $\alpha \equiv 0$, then the boundary condition becomes $\partial_\nu y(x,t) = 0$ and that means there is no heat, gas or population transfer through the boundary $\partial\Omega$.

The main goal of this paper is to precise if for any $y_0 \in L^\infty(\Omega)$, $y_0(x) \geq 0$ a.e. $x \in \Omega$, there exists a control $u \in L^\infty_{loc}(\overline{\omega} \times [0,+\infty))$ such that the solution y^u to (1) satisfies

$$y^u(x,t) \geq 0 \quad a.e.\ x \in \Omega,\ \forall t \geq 0 \tag{2}$$

and

$$\lim_{t \to +\infty} y^u(t) = 0 \quad in\ L^\infty(\Omega). \tag{3}$$

Definition 1. If the answer to the above mentioned question is affirmative, then we say that (1) is *nonnegatively stabilizable* and the property is called *nonnegative stabilizability.*

This is in fact null stabilizability with state constraints. In this case it is also desirable to find a feedback control u that realizes (2) and (3).

We shall also analyze the problem of maximizing of $\lambda_1^{\omega,\Omega}$ with respect to some geometric properties of ω and Ω.

2. Characterization of the nonnegative stabilizability for (1).

For any $\gamma > 0$ we denote by $\lambda_{1\gamma}$ the principal eigenvalue for the following problem:

$$\begin{cases} -div(a(x)\nabla\varphi(x)) + c\varphi(x) + \gamma m\varphi(x) = \lambda\varphi(x), & x \in \Omega \\ \partial_\nu\varphi(x) + \alpha(x)\varphi(x) = 0, & x \in \partial\Omega \end{cases} \tag{4}$$

(here $\gamma \in \mathbf{R}$). We also denote by $\lambda_1^{\omega,\Omega}$, the principal eigenvalue to the following problem

$$\begin{cases} -div(a(x)\nabla\varphi(x)) + c(x)\varphi(x) = \lambda\varphi(x), & x \in \Omega \setminus \overline{\omega} \\ \varphi(x) = 0, & x \in \partial\omega \\ \partial_\nu\varphi(x) + \alpha(x)\varphi(x) = 0, & x \in \partial\Omega. \end{cases}$$

Lemma 1.

$$\lambda_{1\gamma} \nearrow \lambda_1^{\omega,\Omega}$$

as $\gamma \to +\infty$.

The proof is based on Rayleigh's principle.

Theorem 1. *If (1) is nonnegatively stabilizable, then the principal eigenvalue for the above mentioned problem satisfies*

$$\lambda_1^{\omega,\Omega} > 0.$$

Conversely, if $\lambda_1^{\omega,\Omega} > 0$, *then (1) is nonnegatively stabilizable and for* $\gamma > 0$ *large enough, the feedback control* $u := -\gamma y$ *realizes (2) and (3).*

Proof. If (1) is nonnegatively stabilizable, then for any $y_0 \in L^\infty(\Omega)$, $y_0(x) \geq 0$ a.e. $x \in \Omega$, there exists $u \in L^\infty_{loc}(\overline{\omega} \times [0,+\infty))$ such that

$$y^u(x,t) \geq 0 \quad a.e.\ x \in \Omega,\ \forall t \geq 0$$

and

$$\lim_{t\to+\infty} y^u(t) = 0 \quad in\ L^\infty(\Omega).$$

The last relation implies that

$$\lim_{t\to+\infty} y^u(t) = 0 \quad in\ L^2(\Omega).$$

Since $y^u \geq 0$ on $\partial\omega \times [0, +\infty)$, we may conclude by the comparison principle for parabolic operators that

$$y^u(x,t) \geq z(x,t) \geq 0 \quad a.e.\ x \in \Omega \setminus \overline{\omega},\ \forall t \geq 0, \tag{5}$$

where z is the solution to the following problem:

$$\begin{cases} z_t(x,t) - div_x(a(x)\nabla_x z(x,t)) + c(x)z(x,t) = 0, & x \in \Omega \setminus \overline{\omega},\ t > 0 \\ z(x,t) = 0, & x \in \partial\omega,\ t > 0 \\ \partial_\nu z(x,t) + \alpha(x)z(x,t) = 0, & x \in \partial\Omega,\ t > 0 \\ z(x,0) = y_0(x), & x \in \Omega \setminus \overline{\omega}. \end{cases}$$

By (5) we get that

$$\lim_{t\to+\infty} z(t) = 0 \quad in\ L^2(\Omega \setminus \overline{\omega})$$

and consequently we conclude that $\lambda_1^{\omega,\Omega} > 0$.

Conversely, if $\lambda_1^{\omega,\Omega} > 0$, then by Lemma 1 we obtain that for $\gamma > 0$ large enough we have $\lambda_{1\gamma} > 0$. For such $\gamma > 0$ we consider the feedback control $u := -\gamma y$. Problem (1) becomes:

$$\begin{cases} y_t(x,t) - div_x(a(x)\nabla_x y(x,t)) + c(x)y(x,t) & \\ \qquad +\gamma m(x)y(x,t) = 0 & x \in \Omega,\ t > 0, \\ \partial_\nu y(x,t) + \alpha(x)y(x,t) = 0, & x \in \partial\Omega,\ t > 0 \\ y(x,0) = y_0(x), & x \in \Omega \end{cases} \tag{6}$$

and let y be the solution to (6). It is obvious that y satisfies

$$\frac{1}{2} \cdot \frac{d}{dt}(\|y(t)\|^2_{L^2(\Omega)}) + \lambda_{1\gamma}\|y(t)\|^2_{L^2(\Omega)} \leq 0, \quad for\ t \geq 0$$

which implies that

$$\|y(t)\|^2_{L^2(\Omega)} \leq e^{-2\lambda_{1\gamma}t}\|y_0\|^2_{L^2(\Omega)}, \quad \forall\ t \geq 0,$$

and equivalently

$$\|y(t)\|_{L^2(\Omega)} \leq e^{-\lambda_{1\gamma}t}\|y_0\|_{L^2(\Omega)}, \quad \forall t \geq 0.$$

The parabolic $L^2 \to L^\infty$ inequality

$$\|y(t+s)\|_{L^\infty(\Omega)} \leq Ct^{-N/4}\|y(s)\|_{L^2(\Omega)},$$

for any $t > 0$ small enough (where $C > 0$ is a constant) follows in a standard manner and implies that

$$\|y(t)\|_{L^\infty(\Omega)} \leq \tilde{C}e^{-\lambda_{1\gamma}t}\|y_0\|_{L^2(\Omega)}, \quad \forall t \geq 0$$

(where $\tilde{C} > 0$ is a constant). □

Remark 2. The last inequality shows the great importance of finding the position of ω which maximizes the principal eigenvalue $\lambda_1^{\omega,\Omega}$. A greater value for $\lambda_1^{\omega,\Omega}$ leads to a greater convergence rate to zero of the solution to (6).

In case of nonnegative stabilizability, $u := -\gamma y$ is a stabilizing control for (1).

3. The derivative of $\lambda_1^{\omega,\Omega}$ with respect to translations.

We have already remarked that it is an important task to maximize $\lambda_1^{\omega,\Omega}$ subject to the position of ω. We shall investigate here this problem but only subject to translations of ω. Namely, we intend to evaluate the derivative of $\lambda_1^{\omega,\Omega}$ with respect to translations.

Let $\tilde{\omega}$ be a nonempty open subset of $\mathbf{R}^N$ with smooth boundary and consider O the set of all translations ω of $\tilde{\omega}$ satisfying $\omega \subset\subset \Omega$.

For any $\omega \in O$ and $V \in \mathbf{R}^N$ we define the derivative $d\lambda_1^{\omega}(V)$:

$$d\lambda_1^{\omega,\Omega}(V) = \lim_{\varepsilon \searrow 0} \frac{\lambda_1^{\varepsilon V+\omega,\Omega} - \lambda_1^{\omega,\Omega}}{\varepsilon}.$$

Theorem 2. *For any $\omega \in O$ and $V^0 = (V_1^0, V_2^0, ..., V_N^0) \in \mathbf{R}^N$ we have*

$$d\lambda_1^{\omega,\Omega}(V^0) = -\int_{\partial\omega} (a(x)\nabla\varphi^\omega(x))\cdot\nabla\varphi^\omega(x)(V^0\cdot\nu(x))d\sigma,$$

where φ^ω is the eigenfunction corresponding to $\lambda_1^{\omega,\Omega}$ and satisfying $\|\varphi^\omega\|_{L^2(\Omega\setminus\overline{\omega})} = 1$, $\varphi^\omega(x) > 0$ a.e. $x \in \Omega \setminus \overline{\omega}$ and $\nu(x)$ is the normal outward versor at $x \in \partial\omega$ (outward with respect to $\Omega \setminus \overline{\omega}$).

The proof follows in a standard manner.

4. **Evaluations of $\lambda_1^{\omega,\Omega}$.**

It is obvious that in view of stabilizing parabolic equations it is of great importance to maximize $\lambda_1^{\omega,\Omega}$ with respect to the position of ω and to the shape of Ω. We will investigate some of these aspects for the particular case when:

$$\begin{cases} a(x) = I_N, \quad \forall x \in \overline{\Omega} \\ \alpha \equiv 0 \\ c \equiv 0. \end{cases}$$

So, $\lambda_1^{\omega,\Omega}$ is the principal eigenvalue for the following problem

$$\begin{cases} -\Delta\varphi(x) = \lambda\varphi(x), & x \in \Omega \setminus \overline{\omega} \\ \varphi(x) = 0, & x \in \partial\omega \\ \dfrac{\partial\varphi}{\partial\nu}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{7}$$

Theorem 3. *Assume that φ^* is an eigenfunction for (7) corresponding to $\lambda_1^{\omega,\Omega}$, that satisfies in addition:*

$$\begin{cases} 0 < \varphi^*(x) < M, & \forall x \in \Omega \setminus \overline{\omega} \\ \varphi^*(x) = M, & \forall x \in \partial\Omega, \end{cases} \tag{8}$$

where $M > 0$ is a constant. Then we have that

$$\lambda_1^{\omega,\Omega} > \lambda_1^{\omega,\tilde{\Omega}},$$

for any domain $\tilde{\Omega} \subset \mathbf{R}^N$ with smooth boundary and such that $\omega \subset\subset \tilde{\Omega}$, $meas(\tilde{\Omega}) = meas(\Omega)$ and $\tilde{\Omega} \neq \Omega$.

Remark 3. If ω and Ω are balls with the same center, then such a function exists.

Proof. It is obvious that $\lambda_1^{\omega,\Omega} > 0$. Let $\tilde{\Omega} \subset \mathbf{R}^N$ be a domain with smooth boundary such that $\omega \subset\subset \tilde{\Omega}$, $\tilde{\Omega} \neq \Omega$ and $meas(\tilde{\Omega}) = meas(\Omega)$.

Due to Rayleigh's principle we have

$$\lambda_1^{\omega,\Omega} = Min\{\int_{\Omega\setminus\overline{\omega}} |\nabla\varphi|^2 dx;\ \varphi \in H^1(\Omega\setminus\overline{\omega}),\ \|\varphi\|_{L^2(\Omega\setminus\overline{\omega})} = 1,\ \varphi = 0 \text{ on } \partial\omega\}$$

and this minimum is attained for φ^*. Consider the function $\tilde{\varphi}$ given by

$$\tilde{\varphi}(x) = \begin{cases} \varphi^*(x), & x \in \Omega \cap \tilde{\Omega} \setminus \overline{\omega} \\ M, & x \in \tilde{\Omega} \setminus \Omega. \end{cases}$$

This yields

$$\tilde{\varphi} \in H^1(\tilde{\Omega} \setminus \overline{\omega}) \text{ and } \tilde{\varphi} = 0 \text{ on } \partial\omega.$$

On the other hand we have

$$\|\tilde{\varphi}\|_{L^2(\tilde{\Omega}\setminus\overline{\omega})} > \|\varphi^*\|_{L^2(\Omega\setminus\overline{\omega})}$$

and

$$\|\nabla\tilde{\varphi}\|_{L^2(\tilde{\Omega}\setminus\overline{\omega})} \leq \|\nabla\varphi^*\|_{L^2(\Omega\setminus\overline{\omega})}.$$

This inequality yields:

$$\frac{\int_{\tilde{\Omega}\setminus\overline{\omega}} |\nabla\tilde{\varphi}|^2 dx}{\int_{\Omega\setminus\overline{\omega}} |\tilde{\varphi}|^2 dx} < \frac{\int_{\Omega\setminus\overline{\omega}} |\nabla\varphi^*|^2 dx}{\int_{\Omega\setminus\overline{\omega}} |\varphi^*|^2 dx} = \lambda_1^{\omega,\Omega}.$$

Using once again Rayleigh's principle we get that

$$\lambda_1^{\omega,\tilde{\Omega}} < \lambda_1^{\omega,\Omega}.$$

□

Remark 4. If there exists φ^* an eigenfunction of (7) corresponding to $\lambda_1^{\omega,\Omega}$ and satisfying (8), then we may conclude that

$$\begin{aligned} \lambda_1^{\omega,\Omega} &= Max\ \{\lambda_1^{\omega,\tilde{\Omega}};\ \tilde{\Omega} \subset \mathbf{R}^N \ is\ a\ domain\ with\ smooth\ boundary\ and \\ &\qquad satisfying\ \omega \subset\subset \tilde{\Omega},\ meas(\tilde{\Omega}) = meas(\Omega)\} \\ &= Max\ \{\lambda_1^{\tilde{\omega},\Omega};\ \tilde{\omega} \subset\subset \Omega\ is\ an\ isometric\ transform\ of\ \omega\}. \end{aligned}$$

Remark 5. If ω is a ball $\omega \subset\subset \Omega$, then we may conclude by Theorem 3 that

$$\lambda_1^{\omega,\Omega} \leq \lambda_1^{\omega,B},$$

where B is a ball such that

$$meas(B) = meas(\Omega)$$

and

$$B\ and\ \omega\ have\ the\ same\ center.$$

Moreover, we have equality only for $\Omega = B$ and we conclude that the maximal value for $\lambda_1^{\omega,\Omega}$, subject to all domains $\Omega \subset \mathbf{R}^N$ with smooth boundary and satisfying $\omega \subset\subset \Omega$ and $meas(\Omega) = L$, is attained for the ball B of measure L and with the same center as ω.

It is possible to prove that the following Poincaré inequality holds:

$$\int_{\Omega\setminus\overline{\omega}} |\varphi(x)|^2 dx \leq \gamma \int_{\Omega\setminus\overline{\omega}} |\nabla\varphi(x)|^2 dx, \tag{9}$$

$\forall \varphi \in H^1(\Omega \backslash \overline{\omega})$ with $\varphi = 0$ on $\partial\omega$ (here $\gamma > 0$ is a constant independent of φ).

Relation (9) implies that

$$\frac{1}{\gamma} \le \frac{\int_{\Omega\backslash\overline{\omega}} |\nabla\varphi|^2 dx}{\int_{\Omega\backslash\overline{\omega}} |\varphi|^2 dx},$$

$\forall \varphi \in H^1(\Omega \setminus \overline{\omega})$ such that $\varphi \ne 0_{L^2(\Omega\backslash\overline{\omega})}$, $\varphi = 0$ on $\partial\omega$, and this implies that

$$\frac{1}{\gamma} \le \lambda_1^{\omega,\Omega}.$$

It is important to notice that γ depends on the "distance" between $\overline{\omega}$ and $\partial\Omega$. If this "distance" is small, then γ is small and $\frac{1}{\gamma}$ and $\lambda_1^{\omega,\Omega}$ are large.

So, if we wish to find a position of ω for which $\lambda_1^{\omega,\Omega}$ to be great, then it is important to find a position of $\omega \subset\subset \Omega$ for which the "distance" between $\overline{\omega}$ and $\partial\Omega$ to be small.

Remark 6. In practice it is important to precise how "close" to $\partial\Omega$ should we take $\overline{\omega}$ in order to obtain a desired value $\delta > 0$ for the principal eigenvalue for (7). For this reason we may consider the following Cauchy problem:

$$\begin{cases} -\Delta\varphi(x) = \delta\varphi(x) & \\ \varphi(x) = 1, & x \in \partial\Omega \\ \dfrac{\partial\varphi}{\partial\nu}(x) = 0, & x \in \partial\Omega \end{cases}$$

(which has of course a local solution, which in addition satisfies $\varphi(x) < 1$, $\forall x \notin \partial\Omega$).

If there exists $\omega \subset\subset \Omega$ such that $\varphi(x) = 0$ on $\partial\omega$, then it follows by the maximum principle for elliptic operators that $\varphi(x) > 0$, $\forall x \in \Omega \setminus \overline{\omega}$

(if Ω is a ball, then for any $\delta > 0$ such a set ω exists and is a ball with the same center as Ω).

In conclusion, if such an ω exists, then $\lambda_1^{\omega,\Omega} = \delta$ and for any $\tilde{\Omega}$ satisfying $\omega \subset\subset \tilde{\Omega}$ and $meas(\tilde{\Omega}) = meas(\Omega)$, we have

$$\lambda_1^{\omega,\tilde{\Omega}} \leq \delta.$$

Remark 7. If there exists a ball β such that $\omega \subset \eta \subset\subset \Omega$, then by Rayleigh's principle we get that

$$\lambda_1^{\omega,\Omega} \leq \lambda_1^{\eta,\Omega}.$$

By Remark 4 we have that

$$\lambda_1^{\eta,\Omega} \leq \lambda_1^{\eta,B},$$

where B is the ball with the same center as η and $meas(B) = meas(\Omega)$. So, we have that

$$\lambda_1^{\omega,\Omega} \leq \lambda_1^{\eta,B}.$$

This gives an upper bound for $\lambda_1^{\omega,\Omega}$.

5. A predator-prey system.

Let $h(\cdot,t)$ be the spatial density at time t of a prey species distributed over a spatial domain Ω_h in $\mathbf{R}^N$, $N = 1,\ 2$ or 3, and assume its spatio-temporal dynamics is governed by a basic logistic model:

$$\partial_t h - d_1 \Delta h = rh - kh^2, \qquad x \in \Omega_h,\ t > 0,$$

wherein $r > 0$ is the natural growth rate and $k > 0$ is a density dependent effect on mortality due to intraspecific competition within prey.

Let $p(\cdot, t)$ be the spatial density at time t of a predator species distributed over a spatial domain Ω_p in $\mathbf{R}^N$, with $\Omega_h \cap \Omega_p \neq \emptyset$; in absence of the aforementioned prey – assumed to be its unique resource – the predator population will decay at an exponential rate $a > 0$ and its spatio-temporal dynamics is governed by a basic linear model,

$$\partial_t p - d_2 \Delta p = -ap, \qquad x \in \Omega_p, \ t > 0.$$

When both populations are present predation occurs on $\Omega_h \cap \Omega_p$; let us denote by $f(h, p)$ a suitable functional response to predation. The prey dynamics is modified by predation and reads

$$\partial_t h - d_1 \Delta h = rh - kh^2 - \chi(x) f(h, p) p, \qquad (10)$$

$x \in \Omega_h$, $t > 0$, wherein χ is the characteristic function of $\Omega_h \cap \Omega_p$. Prey captured and eaten at time $t > 0$ and location $x' \in \Omega_h \cap \Omega_p$ are transformed into biomass via a conversion factor $\delta > 0$ yielding a numerical response to predation $\delta f(h(x', t), p(x', t))p(x', t)$. We assume that this quantity is distributed over Ω_p via a generic nonnegative kernel $\ell(x, x')$ so that $\delta \ell(x, x') f(h(x', t), p(x', t))p(x', t)$ is the biomass distributed at $x \in \Omega_p$ resulting from predation at $x' \in \Omega_h \cap \Omega_p$. Biomass conservation implies a consistency condition must hold, $\displaystyle\int_{\Omega_p} \ell(x, x')dx = 1$ for each $x' \in \Omega_h \cap \Omega_p$. It is also obvious that $\ell(x, x') = 0$ for each $x' \in \Omega_p \setminus \Omega_h$. In this setting the predator dynamics reads

$$\partial_t p - d_2 \Delta p = -ap + \delta \int_{\Omega_p} \chi(x')\ell(x,x') f(h,p)(x',t) p(x',t)\, dx', \tag{11}$$

$x \in \Omega_p, t > 0$. In applications we have in mind the functional response to predation may take several standard parametric forms, such as Lotka-Volterra, $f(h,p) = \rho h$, Holling type $k+1$, $f(h,p) = \dfrac{\rho h^k}{1+qh^k}$, or Beddington-De Angelis, $f(h,p) = \dfrac{\rho h}{1+qh+sp}$ (with $\rho,\ q,\ s > 0$).

To complete our model boundary conditions must be imposed to both species. We choose no-flux boundary conditions corresponding to isolated populations:

$$\begin{array}{l} \partial_\eta h(x,t) = \nabla h(x,t) \cdot \eta_h(x) = 0, \qquad x \in \partial\Omega_h,\ t > 0, \\ \text{where } \eta_h(x) \text{ denotes a unit normal vector to } \partial\Omega_h \text{ at } x \in \partial\Omega_h, \\ \partial_\eta p(x,t) = \nabla p(x,t) \cdot \eta_p(x) = 0, \qquad x \in \partial\Omega_p,\ t > 0, \\ \text{where } \eta_p(x) \text{ denotes a unit normal vector to } \partial\Omega_p \text{ at } x \in \partial\Omega_p. \end{array} \tag{12}$$

Last nonnegative and bounded initial conditions are prescribed at time $t = 0$:

$$\begin{array}{ll} h(x,0) = h_0(x) \geq 0, & x \in \Omega_h, \\ p(x,0) = p_0(x) \geq 0, & x \in \Omega_p. \end{array} \tag{13}$$

Then (10)-(11)-(12) and (13) is a basic model for our predator-prey system.

Going back to control problems two strategies are devised and investigated.

In order to directly control the predator species one may select an open subdomain, ω with $\overline{\omega} \subset \Omega_p$, and introduce a control, u, harvesting /

culling predators from ω. Let m be the characteristic function of ω. Equation (11) is modified into

$$\partial_t p - d_2 \Delta p = -ap + \delta \int_{\Omega_p} \chi(x')\ell(x,x')f(h,p)(x',t)p(x',t)\,dx' \\ +m(x)u(x,t), \tag{14}$$

for $x \in \Omega_p$ and $t > 0$. A first question to address reads: "is-it possible to find such a control u so that the solution (h,p) of (10)-(14)-(12) and (13) remains componentwise nonnegative and satisfies $p(\cdot,t) \to 0$ as $t \to +\infty$?"

A second strategy consists in selecting an open subdomain ω with $\overline{\omega} \subset \Omega_h$, and introduce a control, v, harvesting prey from ω and reducing its population, size. Equation (10) now reads

$$\partial_t h - d_1 \Delta h = rh - kh^2 - \chi(x)f(h,p)p + m(x)v(x,t), \tag{15}$$

$x \in \Omega_h$, $t > 0$, where m is the characteristic function of ω. A second question to address is: "is-it possible to find such a control v so that the solution (h,p) of (15)-(11)-(12) and (13) remains componentwise non-negative and satisfies $p(\cdot,t) \to 0$ as $t \to +\infty$ (and if possible $h(\cdot,t) \not\to 0$ as $t \to +\infty$) ?"

6. Global existence results and stabilization in a control free setting.

6.1. Main notations and assumptions.

(H1) : ω and Ω_s, $s := h, p$ are nonempty bounded domains in $\mathbf{R}^N$, $N \geq 1$, with smooth boundaries $\partial\omega$ and $\partial\Omega_s$, respectively, so that locally each ω, Ω_s lies on one side of $\partial\omega$ and $\partial\Omega_s$, respectively; $\eta_s(x)$ is a unit normal vector to $\partial\Omega_s$ at x.
Let χ be the characteristic function of the set $\Omega_h \cap \Omega_p$.

(H2) : All coefficients – d_1, r, k, d_2, a and δ – are positive constants. Next $\ell : \Omega_p \times \Omega_p \to [0, +\infty)$ is a measurable and bounded function satisfying

$$\begin{aligned} &\int_{\Omega_p} \ell(x, x')dx = 1, \qquad x' \in \Omega_h, \\ &\ell(x, x') = 0 \qquad \text{a.e. in } \ \Omega_p \times (\Omega_p \setminus \Omega_h). \end{aligned} \tag{16}$$

Last $f : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is locally Lipschitz continuous, $h \mapsto f(h, p)$ being nondecreasing on $[0, +\infty)$ for any $p \geq 0$, $p \mapsto f(h, p)$ being nonincreasing on $[0, +\infty)$ for any $h \geq 0$; $f(0, p) = 0$, $\forall p \in [0, +\infty)$ and $f(h, 0) > 0$, $\forall h \in (0, +\infty)$.

(H3) : $h_0 \in C(\overline{\Omega}_h)$ and $p_0 \in C(\overline{\Omega}_p)$ are nonnegative;
$\|h_0\|_{L^\infty(\Omega_h)} > 0$, $\|p_0\|_{L^\infty(\Omega_p)} > 0$.

Set

$$K^* = \max(K, \|h_0\|_{L^\infty(\Omega_h)}), \text{ where } K = \frac{r}{k},$$

$$M^* = \|\ell\|_{L^\infty(\Omega_p\times\Omega_p)} f(K^*, 0) \ \max\Big(\|p_0\|_{L^1(\Omega_p)} + \delta\|h_0\|_{L^1(\Omega_h)}, \frac{\delta r}{a} K^* |\Omega_h|\Big),$$

wherein $|\Omega|$ is the N-dimensional Lebesgue measure of $\Omega \subset \mathbf{R}^N$.

6.2. Existence results.

$$\partial_t h - d_1 \Delta h = rh - kh^2 - \chi(x) f(h, p)p - m_h(x)h(x, t), \tag{17}$$

$x \in \Omega_h$, $t > 0$ and

$$\partial_t p - d_2 \Delta p = \begin{array}{l} -ap + \delta \displaystyle\int_{\Omega_p} \chi(x')\ell(x,x') f(h,p)(x',t) p(x',t)\, dx' \\ -m_p(x) p(x,t), \end{array} \tag{18}$$

for $x \in \Omega_p$ and $t > 0$, together with the boundary and initial conditions in (12)-(13).

Theorem 4. *Let $m_h \in L^\infty(\Omega_h)$ and $m_p \in L^\infty(\Omega_p)$ be nonnegative. Then problem (17)-(18)-(12) and (13) has a unique componentwise nonnegative and global strong solution. Moreover*

$$\begin{array}{ll} 0 \le h(x,t) \le K^*, & x \in \Omega_h,\ t > 0, \\ 0 \le p(x,t) \le \max(\|p_0\|_{L^\infty(\Omega_p)}, \dfrac{\delta M^*}{a}), & x \in \Omega_p,\ t > 0. \end{array} \tag{19}$$

6.3. **A threshold for predator extinction without control.** Concerning predator extinction in the original system one gets

Proposition 1. *Let λ_1^p be the principal eigenvalue of the problem*

$$\begin{cases} -d_2\Delta\psi + a\psi - \delta \displaystyle\int_{\Omega_p} \chi(x')\ell(x,x')f(K,0)\psi(x')\,dx' = \lambda\psi, \\ \qquad\qquad\qquad\qquad x \in \Omega_p, \\ \partial_\eta \psi(x) = 0, \qquad\quad x \in \partial\Omega_p. \end{cases}$$

Let (h,p) be a strong solution to (10)-(11)-(12) and (13). Then

(i): *when $\lambda_1^p > 0$, then $\lim_{t\to+\infty} p(\cdot,t) = 0$ in $L^\infty(\Omega_p)$ and $\lim_{t\to+\infty} h(\cdot,t) = K$ in $L^\infty(\Omega_h)$;*

(ii): *when $\lim_{t\to+\infty} p(\cdot,t) = 0$ in $L^\infty(\Omega_p)$, then $\lambda_1^p \geq 0$ and $\lim_{t\to+\infty} h(\cdot,t) = K$ in $L^\infty(\Omega_h)$.*

7. **The p-zero stabilization of the predator population.** Assume in addition that $\overline{\omega} \subset \Omega_p$ and $\Omega_p \setminus \overline{\omega}$ is a domain and let m be the characteristic function of ω.

Let $\lambda_1^{\omega,p} \in \mathbf{R}$ be the principal eigenvalue for the elliptic problem:

$$\begin{cases} -d_2\Delta\psi(x) + a\psi(x) - \delta f(K,0) \displaystyle\int_{\Omega_p\setminus\overline{\omega}} \chi(x')\ell(x,x')\psi(x')dx' \\ \qquad\qquad\qquad = \lambda\psi(x), \qquad x \in \Omega_p \setminus \overline{\omega}, \\ \psi(x) = 0, \qquad x \in \partial\omega \\ \partial_\eta\psi(x) = 0, \qquad x \in \partial\Omega_p. \end{cases} \tag{20}$$

Given $\gamma \geq 0$, let $\lambda_{1\gamma}^{\omega,p} \in \mathbf{R}$ be the principal eigenvalue for the elliptic problem:

$$\begin{cases} -d_2\Delta\psi(x) + a\psi(x) - \delta f(K,0) \displaystyle\int_{\Omega_p} \chi(x')\ell(x,x')\psi(x')dx' \\ \qquad\qquad +m(x)\gamma\psi(x) = \lambda\psi(x), \quad x \in \Omega_p, \\ \partial_\eta\psi(x) = 0, \qquad x \in \partial\Omega_p. \end{cases} \tag{21}$$

The existence of both $\lambda_1^{\omega,p}$ and $(\lambda_{1\gamma}^{\omega,p})_{\gamma\geq 0}$ follow from the assertions in the last section.

Lemma 2. *The mapping $\gamma \mapsto \lambda_{1\gamma}^{\omega,p}$ is increasing and continuous; in addition*

$$\lim_{\gamma\to+\infty} \lambda_{1\gamma}^{\omega,p} = \lambda_1^{\omega,p}.$$

Definition 2. The predator population is p-zero stabilizable if for any (h_0, p_0) satisfying **(H3)**, there exists a control $u \in L^\infty_{loc}(\overline{\omega} \times [0,+\infty))$

such that the solution (h, p) to (10)-(14)-(12) and (13) satisfies

$$\begin{cases} h(x,t) \geq 0 & a.e.\ x \in \Omega_h,\ \forall t \geq 0 \\ p(x,t) \geq 0 & a.e.\ x \in \Omega_p,\ \forall t \geq 0 \end{cases} \tag{22}$$

and

$$\lim_{t \to +\infty} p(\cdot, t) = 0 \text{ in } L^\infty(\Omega_p). \tag{23}$$

Hence "p-zero stabilizable" means that the zero stabilizability holds for controls acting only on the predator population.

We now state the main result of this section:

Theorem 5. *If the predator population is p-zero stabilizable then $\lambda_1^{\omega,p} \geq 0$.*

Conversely, when $\lambda_1^{\omega,p} > 0$ the predator population is p-zero stabilizable and for γ large enough the feedback control $u := -\gamma p$ realizes (22) and (23), where (h,p) is the solution to (10)-(14)-(12)-(13), corresponding to $u := -\gamma p$.

Remark 8. This yields $\lambda_1^{\omega,p} > \lambda_1^p$ (the proof follows in the same manner). Theorem 5 implies that even when $\lambda_1^p \leq 0$, p-zero stabilizability holds as soon as $\lambda_1^{\omega,p} > 0$.

Remark 9. When $\varepsilon \longrightarrow 0+$ and $\gamma \longrightarrow +\infty$ we have $\lambda_{1\gamma}^{\omega,p}(\varepsilon) \longrightarrow \lambda_1^{\omega,p}$. This shows the importance of maximizing $\lambda_1^{\omega,p}$ with respect to the position and geometry of ω.

8. **The h-zero stabilization of the predator population.** Assume in addition that $\overline{\omega} \subset \Omega_h$ and $\Omega_h \setminus \overline{\omega}$ is a domain and let m be the characteristic function of ω.

Definition 3. The predator population is h-zero stabilizable if for any (h_0, p_0) satisfying **(H3)** there exists a $v \in L^\infty_{loc}(\overline{\omega} \times [0, +\infty))$ such that the solution (h, p) to (15)-(11)-(12) and (13) remains componentwise nonnegative, cf. (22), and satisfies (23).

"h-zero stabilizable" means that the zero-stabilizability holds for controls acting only on the prey population.

Given any $\gamma > 0$ let (h_γ, p_γ) be the nonnegative solution to

$$\begin{cases} \partial_t h - d_1 \Delta h = rh - kh^2 - \chi(x) f(h,p)p - m(x)\gamma h, \\ \qquad\qquad x \in \Omega_h,\ t > 0, \\ \partial_t p - d_2 \Delta p = -ap + \delta \displaystyle\int_{\Omega_p} \chi(x')\ell(x,x') f(h,p)(x',t) p(x',t) dx', \\ \qquad\qquad x \in \Omega_p, t > 0, \\ \partial_\eta h(x,t) = 0, \qquad x \in \partial\Omega_h,\ t > 0, \\ \partial_\eta p(x,t) = 0, \qquad x \in \partial\Omega_p,\ t > 0, \\ h(x,0) = h_0(x), \qquad x \in \Omega_h, \\ p(x,0) = p_0(x), \qquad x \in \Omega_p. \end{cases} \tag{24}$$

Let now $\mu_1^{\omega,h}$ be the principal eigenvalue to the elliptic problem:

$$\begin{cases} -d_1\Delta\varphi - r\varphi = \mu\varphi, & x \in \Omega_h \setminus \overline{\omega}, \\ \varphi(x) = 0, & x \in \partial\omega, \\ \partial_\eta\varphi(x) = 0, & x \in \partial\Omega_h. \end{cases}$$

Proposition 2. *Assume that* $\mu_1^{\omega,h} > 0$. *Then for large* γ *any solution* (h_γ, p_γ) *to (24) satisfies* $h_\gamma(\cdot,t) \to 0$ *in* $L^\infty(\Omega_h)$ *and* $p_\gamma(\cdot,t) \to 0$ *in* $L^\infty(\Omega_p)$ *as* $t \to +\infty$.

Remark 10. When $\mu_1^{\omega,h} > 0$, using the feedback control $v := -\gamma h$ – for $\gamma \geq 0$ large enough – one gets extinction for both populations and $\lim_{t\to+\infty} p(\cdot,t) = 0$ in $L^2(\Omega_p)$ and in $L^\infty(\Omega_p)$ at an at least exponential rate.

For any $\gamma > 0$ let $\mu_{1\gamma}^{\omega,h}$ be the principal eigenvalue to the elliptic problem:

$$\begin{cases} -d_1\Delta\varphi(x) - r\varphi(x) + m(x)\gamma\varphi(x) = \mu\varphi(x), & x \in \Omega_h \\ \partial_\eta\varphi(x) = 0, & x \in \partial\Omega_h. \end{cases} \tag{25}$$

We now look for h-stabilization conditions preserving the prey population to the expanse of a suitable depletion in spatial density.
For any $\gamma > 0$ let again $\mu_{1\gamma}^{\omega,h}$ be the principal eigenvalue to the elliptic problem (25). The mapping $\gamma \mapsto \mu_{1\gamma}^{\omega,h}$ is increasing, continuous on $[0, +\infty)$ and satisfies $\mu_{10}^{\omega,h} = -r < 0$ and $\lim_{\gamma\to+\infty} \mu_{1\gamma}^{\omega,h} = \mu_1^{\omega,h}$. It follows that there exists a nonnegative $\tilde{\gamma} \geq 0$ – that can possibly be $+\infty$ – such that

$$\begin{array}{l} \mu_{1\gamma}^{\omega,h} > 0,\ for\ \gamma > \tilde{\gamma}\ (if\ \tilde{\gamma} < +\infty) \\ \mu_{1\gamma}^{\omega,h} < 0,\ for\ 0 \leq \gamma < \tilde{\gamma}. \end{array}$$

Let us now choose both ω and γ such that $\mu_{1\gamma}^{\omega,h} < 0$, that is $0 \leq \gamma < \tilde{\gamma}$. Let K_γ be the unique nontrivial and nonnegative solution, $0 < K_\gamma(x) \leq K$ for $x \in \Omega_h$, to the following semilinear boundary value problem

$$\begin{cases} -d_1\Delta K_\gamma = rK_\gamma - kK_\gamma^2 - m(x)\gamma K_\gamma & in\ \Omega_h, \\ \partial_\eta K_\gamma = 0 & in\ \partial\Omega_h. \end{cases}$$

Such a solution K_γ exists and is unique provided $\mu_{1\gamma}^{\omega,h} < 0$.

Last, let $\nu_1^{\omega,\gamma}$ be the principal eigenvalue of the problem

$$\begin{cases} -d_2\Delta\psi + a\psi - \delta \displaystyle\int_{\Omega_p} \chi(x')\ell(x,x')f(K_\gamma(x'),0)\psi(x')dx' = \lambda\psi \\ \qquad in\ \Omega_p, \\ \partial_\eta\psi = 0 \qquad in\ \partial\Omega_p. \end{cases}$$

Theorem 6. *Let* ω *and* γ *be such that* $\mu_{1\gamma}^{\omega,h} < 0$. *Then the predator population is h-zero stabilizable and the feedback control* $v := -\gamma h$ *realizes (22) and (23), where (h,p) is the solution to (24). Moreover* $\lim_{t\to+\infty} h(\cdot,t) = K_\gamma$ *in* $L^\infty(\Omega_h)$.

9. **Final comments.**

The analysis of the first stabilizing strategy shows the importance of finding the position and the geometry of ω that maximizes $\lambda_1^{\omega,p}$.

As concern the second strategy, we begin by emphasizing that the mapping $\gamma \mapsto \mu_{1\gamma}^{\omega,h}$ is increasing and continuous on $[0,+\infty)$ and $\mu_{10}^{\omega,h} = -r < 0$ and $\lim_{\gamma\to+\infty} \mu_{1\gamma}^{\omega,h} = \mu_1^{\omega,h}$.

It is possible to prove that the mapping $\gamma \mapsto K_\gamma$ is nonincreasing and continuous from $[0,+\infty)$ to $L^\infty(\Omega_h)$. This implies that the mapping $\gamma \mapsto \nu_1^{\omega,\gamma}$ is nondecreasing and continuous on $[0,+\infty)$ and $\nu_1^{\omega,0} = \lambda_1^p$.

The results concerning the second stabilizing strategy show the importance of finding the position and the geometry of ω for which exists a $\gamma \geq 0$ such that $\mu_{1\gamma}^{\omega,h} < 0$ and $\nu_1^{\omega,\gamma} > 0$.

The idea is here to reduce the density of prey via a harvesting process in ω (with a constant harvesting effort γ) up to a level that cannot sustain anymore the predator population but assures the persistence of preys.

It is important to notice that we could combine both strategies: we could harvest the prey population in the subdomain ω_1 (with a constant harvesting effort γ_1) and harvest the predator population in the subdomain ω_2 (with a harvesting effort γ_2). This way it is possible to stabilize the predator population to 0 even in situations when the previous two strategies fail. Similar stabilizability results can be obtained.

Remark that in case of stabilizability of the predator population (in all situations), the stabilizing controls are simple (of feedback type); they are in fact constant harvesting efforts.

10. **Some auxiliary results.**

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and let η be a unit normal vector to $\partial\Omega$ along Ω. Let $\alpha \in L^\infty(\Omega)$ and $F \in L^\infty(\Omega \times \Omega)$ be such that $F(x, x') \geq 0$ a.e. in $\Omega \times \Omega$ and let $\zeta \in \mathbf{R}$ satisfy

$$\zeta > \|F\|_{L^2(\Omega\times\Omega)} + \|\alpha\|_{L^\infty(\Omega)}.$$

Lax-Milgram's Lemma yields that for any $g \in L^2(\Omega)$ there exists a unique $\mathcal{T}g \in H^1(\Omega)$, a solution to

$$\begin{cases} -\Delta\psi(x) + \alpha(x)\psi(x) - \displaystyle\int_\Omega F(x, x')\psi(x')dx' + \zeta\psi(x) = g(x), \\ \qquad\qquad\qquad x \in \Omega, \\ \partial_\eta\psi(x) = 0, \qquad x \in \partial\Omega. \end{cases} \tag{26}$$

Then, $\mathcal{T} : L^2(\Omega) \to L^2(\Omega)$ is a linear compact operator, with positive spectral radius $\rho(\mathcal{T})$ and satisfies $\mathcal{T}(\mathcal{K}) \subset \mathcal{K}$, where $\mathcal{K} = \{\psi \in L^2(\Omega);\ \psi(x) \geq 0 \ a.e.\ in\ \Omega\}$.

Since $\mathcal{K}$ is a solid – closed and convex – cone, Krein-Rutman's Theorem, yields that the spectral radius $\rho(\mathcal{T})$ of $\mathcal{T}$ is an eigenvalue of $\mathcal{T}$ with an eigenvector $\tilde{\psi} \in \mathcal{K} \setminus \{0\}$, $\mathcal{T}\tilde{\psi} = \rho(\mathcal{T})\tilde{\psi}$. Moreover, $\rho(\mathcal{T}^*) = \rho(\mathcal{T})$ is an eigenvalue of the adjoint operator $\mathcal{T}^*$ with an eigenvector $\tilde{\psi}^* \in \mathcal{K} \setminus \{0\}$.

As a consequence $\lambda_1 = \dfrac{1}{\rho(\mathcal{T})} - \zeta$ is an eigenvalue for the elliptic problem

$$\begin{cases} -\Delta\psi(x) + \alpha(x)\psi(x) - \displaystyle\int_\Omega F(x,x')\psi(x')dx' = \lambda\psi(x), & x \in \Omega, \\ \partial_\eta \psi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{27}$$

with $\tilde{\psi}$ as a corresponding nonnegative eigenfunction, and λ_1 is also an eigenvalue for the adjoint problem

$$\begin{cases} -\Delta\psi^*(x) + \alpha(x)\psi^*(x) - \displaystyle\int_\Omega F(x',x)\psi^*(x')dx' = \lambda\psi^*(x), \\ \qquad\qquad x \in \Omega, \\ \partial_\eta \psi^*(x) = 0, \qquad x \in \partial\Omega, \end{cases} \tag{28}$$

Let us now check that when $g \in \mathcal{K} \setminus \{0\}$, then $\psi = \mathcal{T}g \in Int(\mathcal{K})$. Indeed, when $g \in L^2(\Omega)$, $g(x) \geq 0$ a.e. Ω, $g \not\equiv 0$, and $\psi(x) \geq 0$ a.e. in Ω , $\psi \not\equiv 0$, it follows from the strong maximum principle for elliptic operators that $\psi = \mathcal{T}g$ the solution to (26) is positive in Ω and $\psi \in Int(\mathcal{K})$.

A consequence of Krein-Rutman's Theorem says that λ_1 is a simple eigenvalue for (27) and (28) and that there is no other eigenvalue with positive eigenfunctions.

λ_1 *is called the principal eigenvalue for (27), and for (28) as well.*

Using an approximating technique one may conclude that similar results hold for the principal eigenvalue and corresponding eigenfunctions for

$$\begin{cases} -\Delta\psi(x) + \alpha(x)\psi(x) - \int_\Omega F(x,x')\psi(x')dx' = \lambda\psi(x), & x \in \Omega, \\ \psi(x) = 0, & x \in \Gamma, \\ \partial_\eta \psi(x) = 0, & x \in \partial\Omega \setminus \Gamma, \end{cases} \tag{29}$$

wherein Γ is a measurable subset of $\partial\Omega$.

We shall establish now a comparison result.
Set $V = \{q \in H^1(\Omega);\ q = 0 \ on\ \Gamma\}$. Let $p_i \in C([0,T]; L^2(\Omega)) \cap C((0,T]; V) \cap C^1((0,T]; L^2(\Omega))$ $(i = 1,2)$ for any $T \in (0,+\infty)$ be solutions to

$$\begin{cases} \partial_t p_i - d\Delta p_i = -ap_i + \int_\Omega F_i(x,x')p_i(x',t)dx', & x \in \Omega,\ t > 0, \\ p_i(x,t) = \zeta_i(x), & x \in \Gamma,\ t > 0, \\ \partial_\eta p_i(x,t) = 0, & x \in \partial\Omega \setminus \Gamma,\ t > 0, \\ p_i(x,0) = p_{0i}(x), & x \in \Omega, \end{cases}$$

where $d > 0$ and $a > 0$ (when such solutions exist).

Lemma 3. *(***A comparison result***) Assume in addition that for $i = \overline{1,2}$: $p_{0i} \in L^\infty(\Omega)$, $\zeta_i \in$*

$L^\infty(\partial\Omega)$, $F_i \in L^\infty(\Omega \times \Omega)$ *and*

$$\begin{cases} 0 \le p_{01}(x) \le p_{02}(x) & a.e.\ x \in \Omega, \\ 0 \le \zeta_1(x) \le \zeta_2(x) & a.e.\ x \in \Gamma, \\ 0 \le F_1(x,x') \le F_2(x,x') & a.e.\ (x,x') \in \Omega \times \Omega. \end{cases}$$

Then $0 \le p_1(x,t) \le p_2(x,t)$ *a.e.* $x \in \Omega,\ \forall t \ge 0.$

We may now pass to a second auxiliary result. Let $\alpha_n \in L^\infty(\Omega)$, $F_n \in L^\infty(\Omega \times \Omega)$ $(n \in \mathbf{N})$ such that

$$F_n(x,x') \ge 0 \qquad a.e.\ in\ \Omega \times \Omega.$$

Denote by λ_{1n} the principal eigenvalue for

$$\begin{cases} -\Delta\psi(x) + \alpha_n\psi(x) - \int_\Omega F_n(x,x')\psi(x')dx' = \lambda\psi(x), & x \in \Omega \\ \psi(x) = 0, & x \in \Gamma \\ \partial_\eta\psi(x) = 0, & x \in \partial\Omega \setminus \Gamma \end{cases}$$

and by ψ_n its positive eigenfunction satisfying $\|\psi_n\|_{L^2(\Omega)} = 1$.

Theorem 7. *(***An approximating result***) Assume in addition that*

$$F_n \to F \qquad in\ L^\infty(\Omega \times \Omega),$$
$$\alpha_n \to \alpha \qquad in\ L^\infty(\Omega),$$

as $n \to +\infty$*; then*

$$\lambda_{1n} \to \lambda_1,$$
$$\psi_n \to \tilde{\psi} \qquad in\ L^2(\Omega),$$

as $n \longrightarrow +\infty$, *where* λ_1 *is the principal eigenvalue to (29) and* $\tilde{\psi}$ *is the positive eigenfunction corresponding to* λ_1 *and satisfying* $\|\tilde{\psi}\|_{L^2(\Omega)} = 1$*).*